Data provided: Formula sheet

MAS248

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2012–13**

MATHEMATICS III (CHEMICAL) **2 hours**

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) Find and classify the stationary points of the function

$$f(x, y) = \frac{x^3}{3} - x + \frac{y^2}{2} + 2y.$$

(9 marks)

(ii) Write down the iteration formula for the Newton-Raphson method. Starting from $x_0 = 3.0$ perform three iterations of the Newton-Raphson method to find an approximation to a root of the equation

$$x^3 - 2x^2 - 5 = 0.$$

Work correct to six decimal places throughout. ***(7 marks)***

(iii) Find $\dfrac{dy}{dx}$ given that

$$x \cos 3y + x^3 y^5 = 3x - e^{xy}.$$

(9 marks)

2 (i) Write down the definitions of an odd and of an even function. State whether each of the following functions is odd, even or neither, and whether it is periodic or not periodic:

(a) $f(x) = x^4$;

(b) $f(x) = \sin\left(2x - \dfrac{\pi}{2}\right)$;

(c) $f(x) = x^2 \cos x$.
If the function is periodic state its period. ***(8 marks)***

(ii) A periodic function $f(x)$ of period 2π is defined by

$$f(x) = \begin{cases} 0 & \text{for} \quad -\pi \le x < -\dfrac{\pi}{2}, \\ 4 & \text{for} \quad -\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}, \\ 0 & \text{for} \quad \dfrac{\pi}{2} < x < \pi. \end{cases}$$

Sketch a graph of $f(x)$ for $-3\pi \le x \le 3\pi$. ***(6 marks)***

Find the first four non-zero terms of the Fourier series expansion of $f(x)$. ***(11 marks)***

3 (i) A scalar field, ϕ, is defined by

$$\phi = x^2 y + y^2 z + z^2 x.$$

Find $\nabla \cdot (\nabla \phi)$. ***(6 marks)***

(ii) For the vector

$$\mathbf{F} = (3x^2y^2, 2x^3y + \cos z, -y \sin z),$$

find a scalar potential, V, such that

$$\mathbf{F} = \nabla V.$$

(11 marks)

(iii) Two vector fields, $\mathbf{A}$ and $\mathbf{B}$, are given by

$$\mathbf{A} = (x^2y, 2xz, x)$$

and

$$\mathbf{B} = (yz, xy, 3).$$

For these vectors verify that

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}).$$

(8 marks)

4 Show that the partial differential equation

$$\frac{\partial^2 y}{\partial t^2} = 4\frac{\partial^2 y}{\partial x^2},$$

has solutions of the form $y = f(x + \lambda t)$ for arbitrary functions f provided that $\lambda = -2$ or $\lambda = 2$. ***(6 marks)***

Give an interpretation, including a clear diagram, of the form of the solution in each case. ***(6 marks)***

Derive the solution that satisfies the conditions

$$y(x, 0) = \sin x,$$

$$\frac{\partial y}{\partial t}(x, 0) = e^{-x}.$$

(13 marks)

End of Question Paper

Formula Sheet

Fourier Series

Suppose that $f(x)$ is defined on the interval $-L \leq x \leq L$. The Fourier series for $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right),$$

where

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx, \qquad n = 0, 1, 2, ...,$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx, \qquad n = 1, 2, 3, ...$$

On the interval $0 \leq x \leq L$, the Fourier cosine series for $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}, \qquad a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx,$$

and the Fourier sine series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \qquad b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx.$$

Gradient of a Scalar Field

The gradient of the scalar field, $\phi(x, y, z)$, is given by

$$\nabla\phi = \text{grad } \phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z} \right).$$

Chain Rule

1 If $z = f(x, y)$, where $x = x(t)$, $y = y(t)$, then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

2 If $z = f(x, y)$, where $x = x(u, v)$, $y = y(u, v)$, then

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}, \qquad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}.$$

3 If $z = f(u, v)$, where $u = u(x, y)$, $v = v(x, y)$, then

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}, \qquad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}.$$

Maxima and Minima

1 The function $f(x, y)$ has a stationary point at (x_0, y_0) if

$$f_x = f_y = 0 \quad \text{at} \quad (x_0, y_0).$$

2 At (x_0, y_0), the function $f(x, y)$ has:

(i) a minimum if

$$f_{xx}f_{yy} - f_{xy}^2 > 0 \quad \text{and} \quad f_{xx} > 0 \quad \text{at} \quad (x_0, y_0),$$

(ii) a maximum if

$$f_{xx}f_{yy} - f_{xy}^2 > 0 \quad \text{and} \quad f_{xx} < 0 \quad \text{at} \quad (x_0, y_0),$$

(iii) a saddle point if

$$f_{xx}f_{yy} - f_{xy}^2 < 0 \quad \text{at} \quad (x_0, y_0).$$