Data provided: Formula Sheet

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2012-2013**

Mathematics II (Materials) **2 hours**

Marks will be awarded for answers to all questions in Section A, and for your best ***THREE*** *answers to questions in Section B. Section A carries 40 marks, and the marks awarded to each question or section of question are shown in italics.*

Section A

A1 Find the solution of the equation

$$\frac{dy}{dx} = \frac{y}{x} + \frac{x}{2y}$$

for $x > 0$ which satisfies $y = 1$ when $x = 1$. ***(8 marks)***

A2 Find the general solution of the equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = e^{3x}$$

(8 marks)

A3 Alice throws a ball of mass $0.6\,\text{kg}$ at a speed of $10\,\text{ms}^{-1}$. The kinetic energy of the ball is

$$E = \frac{1}{2}mv^2,$$

where m and v are its mass and velocity, respectively. If Alice now throws a ball of mass $0.56\,\text{kg}$ at a speed of $10.5\,\text{ms}^{-1}$, use the chain rule for partial derivatives to estimate the % change in the kinetic energy she must give the ball. ***(7 marks)***

A4 A scalar field f is given by

$$f = xy + 2yz + 3zx,$$

and a vector field $\mathbf{u}$ is defined by

$$\mathbf{u} = \nabla f\ .$$

(a) Find $\mathbf{u}$ and $\nabla \cdot \mathbf{u}$, and show that $\nabla \times \mathbf{u} = \mathbf{0}$. ***(6 marks)***

(b) Find the directional derivative of f in the direction of the vector $(1, -2, 3)$ at the point $(1, 1, -1)$. ***(5 marks)***

A5 Two quantities x and y have means 13.4 and 95.7 respectively, variances 2.8 and 54.3 respectively, and covariance 11.9.

(a) Calculate the correlation coefficient between x and y, correct to 3 significant figures. ***(2 marks)***

(b) It is assumed that x and y satisfy the linear relationship

$$y = a + b(x - \bar{x}), \qquad (*)$$

where $\bar{x}$ is the mean of x.

Calculate the least squares estimates of a and b, correct to 3 significant figures. State, giving reasons, whether you expect $(*)$ to give a good model. ***(4 marks)***

Section B

B1 (a) Find the general solution of the equation

$$(1 + x^2)\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy = \mathrm{e}^{2x}.$$

(10 marks)

(b) Find the solution of the equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 6\frac{\mathrm{d}y}{\mathrm{d}x} + 9y = 18\mathrm{e}^{2x},$$

. given that $y = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ when $x = 0$.

(10 marks)

B2 (a) If $f = (x^2 + y^2 + z^2)^{-1/2}$, show that

$$\frac{\partial^2 f}{\partial x^2} = \frac{2x^2 - y^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}}.$$

Hence deduce that

$$\nabla^2 f = 0.$$

(12 marks)

(b) A sample of 20 lightbulbs had the following lifetimes (in weeks, rounded to the nearest week):

30, 36, 28, 37, 35, 31, 27, 39, 33, 34, 38, 45, 23, 41, 36, 37, 19, 28, 30, 33

Calculate the mean and standard deviation of the bulb lifetime, correct to 2 decimal places.

(8 marks)

B3 (a) By integrating by parts twice, evaluate

$$\int_0^{\pi} \mathrm{e}^{-x} \sin nx \, \mathrm{d}x,$$

where n is a positive integer. ***(9 marks)***

(b) A function $f(x) = \mathrm{e}^{-x}$ is defined on the interval $0 \le x \le \pi$.

(i) Show that $f(x)$ can be represented by the Fourier sine series

$$\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{n\{1-(-1)^n \mathrm{e}^{-\pi}\}}{1+n^2} \sin nx .$$

(7 marks)

(ii) Sketch the function given by the above Fourier sine series on the interval

$$-2\pi \le x \le 2\pi \; .$$

(4 marks)

B4 Suppose that $u(x,t)$ satisfies the heat conduction equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

on $0 \le x \le l$, for $t > 0$, subject to the boundary conditions

$$u = 0 \text{ at } x = 0 \text{ and at } x = l .$$

(a) Show that the general solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \exp\left(\frac{-n^2\pi^2 t}{l^2}\right) \sin\left(\frac{n\pi x}{l}\right).$$

(12 marks)

(b) Suppose that u also satisfies the initial condition

$$u = \frac{4u_0 x}{l}\left(1 - \frac{x}{l}\right) \qquad \text{for } 0 < x < l$$

at $t = 0$, where u_0 is a constant.

Sketch the initial condition for $0 < x < l$, and determine the constants B_n .

(8 marks)

[You may assume that

$$\int_0^l \frac{x}{l}\left(1 - \frac{x}{l}\right) \sin\left(\frac{n\pi x}{l}\right) \mathrm{d}x = \begin{cases} 0 & n \text{ even} \\ \dfrac{4l}{(n\pi)^3} & n \text{ odd} \end{cases}$$

]

End of Question Paper

FORMULA SHEET

Trigonometry

$1+\tan^2 q = \sec^2 q$

$1+\cot^2 q = \operatorname{cosec}^2 q$

$\cos(A+B)=\cos A\cos B-\sin A\sin B$

$\cos(A-B)=\cos A\cos B+\sin A\sin B$

$\sin(A+B)=\sin A\cos B+\cos A\sin B$

$\sin(A-B)=\sin A\cos B-\cos A\sin B$

$\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}$

$\tan(A-B)=\dfrac{\tan A-\tan B}{1+\tan A\tan B}$

$\sin 2q = 2\sin q\cos q$

$\cos 2q = 2\cos^2 q -1 = 1-2\sin^2 q$

$a\cos q + b\sin q = R\cos(q-a)$, where $R=\sqrt{a^2+b^2}$, $\cos a = a/R$ and $\sin a = b/R$

Hyperbolic Functions

$\sinh x = \dfrac{1}{2}\left(e^x - e^{-x}\right)$

$\cosh x = \dfrac{1}{2}\left(e^x + e^{-x}\right)$

$\cosh^2 x - \sinh^2 x = 1$

$\operatorname{sech}^2 x + \tanh^2 x = 1$

$\sinh 2x = 2\sinh x\cosh x$

$\cosh 2x = 2\cosh^2 x - 1 = 2\sinh^2 x + 1$

$\sinh^{-1} x = \ln\left(x+\sqrt{1+x^2}\right)$, all x

$\cosh^{-1} x = \ln\left(x+\sqrt{x^2-1}\right)$, $x \ge 1$

$\tanh^{-1} x = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$, $|x|<1$

$\coth^{-1} x = \dfrac{1}{2}\ln\left(\dfrac{x+1}{x-1}\right)$, $|x|>1$

Differentiation and Integration

Function	Derivative
x^n	nx^{n-1}
$\ln x$	$\frac{1}{x}$
e^x	e^x
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\operatorname{sech}^2 x$
$\coth x$	$-\operatorname{cosech}^2 x$
$\operatorname{sech} x$	$-\operatorname{sech} x \tanh x$
$\operatorname{cosech} x$	$-\operatorname{cosech} x \coth x$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\cot^{-1} x$	$-\frac{1}{1+x^2}$
$\sinh^{-1} x$	$\frac{1}{\sqrt{x^2+1}}$
$\cosh^{-1} x$	$\frac{1}{\sqrt{x^2-1}}$
$\tanh^{-1} x$	$\frac{1}{1-x^2}, \qquad \lvert x\rvert < 1$
$\coth^{-1} x$	$-\frac{1}{x^2-1}, \qquad \lvert x\rvert > 1$

Function	**Integral**
$\dfrac{1}{a^2+x^2}$	$\dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)$
$\dfrac{1}{a^2-x^2}$	$\dfrac{1}{a}\tanh^{-1}\left(\dfrac{x}{a}\right)$
$\dfrac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\dfrac{x}{a}\right)$
$\dfrac{1}{\sqrt{a^2+x^2}}$	$\sinh^{-1}\left(\dfrac{x}{a}\right)$
$\dfrac{1}{\sqrt{x^2-a^2}}$	$\cosh^{-1}\left(\dfrac{x}{a}\right)$

Differentiation and Integration Formulae

$$\frac{\mathrm{d}(uv)}{\mathrm{d}x}=\frac{\mathrm{d}u}{\mathrm{d}x}v+u\frac{\mathrm{d}v}{\mathrm{d}x}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{v}\right)=\frac{\dfrac{\mathrm{d}u}{\mathrm{d}x}v-u\dfrac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$$

$$\int_a^b uv\,\mathrm{d}x=\left[u\times(\text{integral of } v)\right]_a^b-\int_a^b\frac{\mathrm{d}u}{\mathrm{d}x}\times(\text{integral of } v)\,\mathrm{d}x$$

or $$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x=uv-\int\frac{\mathrm{d}u}{\mathrm{d}x}v\,\mathrm{d}x$$

Partial differentiation

Chain Rule

1. Suppose that $z=f(x,y)$ and that x and y are functions of t, i.e., $x=x(t)$, $y=y(t)$. Then

$$\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}$$

2. Suppose that $z=f(x,y)$ and that x and y are functions of the variables r and s, i.e., $x=x(r,s)$, $y=y(r,s)$. Then

$$\frac{\partial z}{\partial r}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial r},\quad \frac{\partial z}{\partial s}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

First-Order Differential Equations

1. **Direct Integration**

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)$$

$$y = \int f(x)\,\mathrm{d}x + C$$

2. **Separation of Variables**

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(x)$$

$$\int \frac{\mathrm{d}y}{g(x)} = \int f(x)\,\mathrm{d}x$$

3. **Homogeneous Equations**

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f\left(\frac{y}{x}\right)$$

Make the substitution $y = zx$ to give

$$z + x\frac{\mathrm{d}z}{\mathrm{d}x} = f(z)$$

4. **Linear Equations**

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$$

Multiply both sides by the integrating factor $\mathrm{e}^{\int P(x)\,\mathrm{d}x}$ to give

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(y\mathrm{e}^{\int P(x)\,\mathrm{d}x}\right) = Q(x)\mathrm{e}^{\int P(x)\,\mathrm{d}x}$$

The Second-Order Differential Equation

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+b\frac{\mathrm{d}y}{\mathrm{d}x}+cy=f(x)$$

where a, b and c are constants.

The General Solution is

$y =$ Complementary Function + Particular Integral

Complementary Function

If the auxiliary equation $am^2+bm+c=0$ has roots m_1, m_2,

then the Complementary Function y_c is given by

(i) $y_c = A\mathrm{e}^{m_1x}+B\mathrm{e}^{m_2x}$ if m_1 and m_2 are real and different

(ii) $y_c = \mathrm{e}^{mx}(A+Bx)$, if m_1 and m_2 are real and equal $(m_1=m_2=m)$

(iii) $y_c = \mathrm{e}^{px}(A\cos qx+B\sin qx)$, if m_1 and m_2 are complex $(m_1=p+\mathrm{i}q,\ m_2=p-\mathrm{i}q)$

Particular Integral, y_p

$f(x)=Ax^2+Bx+C$ $\qquad y_p=ax^2+bx+c$

$f(x)=A\mathrm{e}^{kx}$ $\qquad y_p=a\mathrm{e}^{kx}$

where k is not one of the roots of the auxiliary equation

$f(x)=A\mathrm{e}^{kx}$ $\qquad y_p=ax\mathrm{e}^{kx}$

where k is one of the roots of the auxiliary equation

$f(x)=A\cos mx+B\sin mx$ $\qquad y_p=a\cos mx+b\sin mx$

where $\sin mx$ or $\cos mx$ is not part of the complementary function

$f(x)=A\cos mx+B\sin mx$ $\qquad y_p=ax\cos mx+bx\sin mx$

where $\sin mx$ or $\cos mx$ is part of the complementary function

Fourier Series

Suppose that $f(x)$ is defined on the interval $-l \le x \le l$. The Fourier series for $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{npx}{l} + b_n \sin \frac{npx}{l} \right),$$

where

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{npx}{l} \, \mathrm{d}x, \qquad n = 0, 1, 2, \mathbf{K}$$

$$b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{npx}{l} \, \mathrm{d}x, \qquad n = 1, 2, \mathbf{K}$$

On the interval $0 \le x \le l$ the Fourier cosine series for $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{npx}{l}, \qquad a_n = \frac{2}{l} \int_{0}^{l} f(x) \cos \frac{npx}{l} \, \mathrm{d}x$$

and the Fourier sine series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{npx}{l}, \quad b_n = \frac{2}{l} \int_{0}^{l} f(x) \sin \frac{npx}{l} \, \mathrm{d}x$$

Vector Calculus

The gradient of the scalar field $f(x, y, z)$ is given by

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

The divergence of a vector field $\mathbf{u}(x, y, z) = (u, v, w)$ is given by

$$\nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

The curl of a vector field $\mathbf{u}(x, y, z) = (u, v, w)$ is given by

$$\nabla \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

The Laplacian ∇^2 is given by

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Statistics

For data values $(x_1, y_1), (x_2, y_2), \mathbf{K}, (x_n, y_n)$

Means $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ etc.

Variances $s_x^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \bar{x}^2$ etc.

Covariance $\operatorname{cov}(x, y) = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) = \frac{1}{n}\sum_{i=1}^{n} x_i y_i - \bar{x}\,\bar{y}$

Correlation coefficient $r = \frac{\operatorname{cov}(x, y)}{s_x s_y}$

Linear Regression by Least Squares

The least squares fit to the linear relationship

$$y = a + b(x - \bar{x})$$

is given by

$$a = \bar{y}, \quad b = \frac{\operatorname{cov}(x, y)}{s_x^2}$$

The corresponding mean square residual is $s_y^2(1 - r^2)$.