The University Of Sheffield.

MAS252

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2012–13**

Further Civil Engineering Mathematics and Computing **2 hours**

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) Show that the Fourier series expansion of the function $f(x) = \exp\left(\frac{x}{2}\right)$, defined in the interval $-\pi \leq x \leq \pi$, is given by

$$\frac{2\sinh\left(\frac{\pi}{2}\right)}{\pi} + \frac{4\sinh\left(\frac{\pi}{2}\right)}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2+1}(\cos nx - 2n\sin nx).$$

Note: $\sinh x = \dfrac{e^x - e^{-x}}{2}$. ***(21 marks)***

By setting $x = 0$, show that

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2+1} = \frac{\pi - 2\sinh\left(\frac{\pi}{2}\right)}{4\sinh\left(\frac{\pi}{2}\right)}.$$

(4 marks)

2 (i) Find the first four non-zero terms of the series solution of the differential equation

$$y'' - \frac{2y}{x} = 0, \quad y = y(x),$$

subject to the conditions $y(1) = 1$ and $y'(1) = 1$. ***(11 marks)***

(ii) Use the chain rule to evaluate the value of dw/dt at $t = 0$ given that $w(r, s, v) = r^2 - s\tan v$ where $r(t) = \sin^2 t$, $s(t) = \cos t$ and $v(t) = 4t$. ***(8 marks)***

(iii) Newton's equation, $x^3 - 2x - 5 = 0$, has a root near $x = 3$. Define the Newton-Raphson formula for calculating the root of a function. Starting with $x_0 = 3$, compute x_1, x_2, and x_3, the next three Newton-Raphson estimates for the root correct to four decimal places. ***(6 marks)***

3 (i) Prove the identity

$$\int_0^{\pi} \sin(nx)\sin(mx) = \begin{cases} 0, & \text{if } m \neq n \\ \pi/2, & \text{if } m = n. \end{cases}$$

(6 marks)

(ii) Use the result of part (i) together with the method of variable separation to find the solution of the heat conduction equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}, \qquad u = u(x,t),$$

subject to the boundary and initial conditions

$$u(0,t) = 0, \qquad \frac{\partial u}{\partial x}(\pi, t) = 0$$

$$u(x,0) = 3\sin\left(\frac{5x}{2}\right) = f(x)$$

(19 marks)

4 (i) Values of $y(x)$ at $x = 2$ determined using the fourth-order Runge-Kutta method in conjunction with an ordinary differential equation with two different step-lengths (h) are given in the following table

h	$y(2)$
0.2	3.40978
0.4	3.39278

Use this data to estimate a value for h which will ensure that the error in the calculated value of $y(2)$ using a fourth-order Runge-Kutta method does not exceed 10^{-4}. Give your answer correct to 4 decimal places.

(7 marks)

(ii) Let us consider the function

$$f(x,y) = xe^{xy}.$$

Find all values of x such that the equality

$$\left.\frac{\partial f(x,y)}{\partial x}\right|_{y=1} = \left.\frac{\partial f(x,y)}{\partial y}\right|_{y=1}$$

holds. ***(6 marks)***

(iii) The temperature, T, measured at the point $P(x,y,z)$ of a steel beam is given in a rectangular coordinate system by

$$T = \left[2x^2 + ln(xy) + 1/z\right]^{1/2}.$$

Use the small error formula to estimate the change in temperature if the measurement is moved from the position $(6,3,2)$ to $(6.1, 3.3, 1.98)$. Work correct to four decimal places.. ***(12 marks)***

End of Question Paper

Formula sheet

- The local truncation error in the case of the 4th order Runge-Kutta method is given by
$$Y(x) - y(x) = Ch^4$$
where $Y(x)$ is the exact value, $y(x)$ is the estimated numerical value, C is a constant and h is the step size used in the numerical scheme.

- **Chain rule**

If $z = f(x, y)$, where x and y are both functions of t, so that $x = x(t)$ and $y = y(t)$ we have
$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$
If $z = f(x, y)$ and both x and y are functions of u and v, so that $x = x(u, v)$ and $y = y(u, v)$ then we have
$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$
$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

- **Fourier series**

If the function $f(x)$ is defined over the interval $-l \leq x \leq l$, then the Fourier series of $f(x)$ is
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$
where
$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\ dx, \quad (n = 0, 1, 2, \dots)$$
$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\ dx \quad (n = 1, 2, 3, \dots)$$
If the function $f(x)$ is defined over the interval $0 \leq x \leq l$, then the Fourier cosine series of $f(x)$ is
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}, \qquad a_n = \frac{2}{l}\int_{0}^{l} f(x)\cos\frac{n\pi x}{l}\ dx, \quad (n = 0, 1, 2, \dots)$$
while the sine series of $f(x)$ is
$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}, \qquad b_n = \frac{2}{l}\int_{0}^{l} f(x)\sin\frac{n\pi x}{l}\ dx \quad (n = 1, 2, 3, \dots)$$

- **Some trigonometric identities**

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1, \quad \sin 2\alpha = 2\sin\alpha\cos\alpha$$