Data provided: Formula sheet

MAS253

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2012-2013**

Mathematics for Engineering Modelling **2 hours**

*Answer **four** questions. If you answer more than four questions, only your best four will be counted.*

1 (i) Derive the Maclaurin series for $\sin x$ and for $\cos x$ each up to the x^4 term.

Given that

$$e^x \approx 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 ,$$

use your approximations to find an expression for $e^{i\theta}$ in terms of i, $\sin\theta$ and $\cos\theta$, where i is the imaginary unit and θ is a real number.

Find the first non-zero term in the Taylor series for $\ln(\sin x)$ about the point $x = \pi/2$.

(8 marks)

(ii) Given the infinite geometric series

$$a + a\,r + a\,r^2 + \ldots + a\,r^{n-1} + \ldots, \qquad r \neq 1,$$

derive an expression for the partial sum s_n of the first n terms. Hence show that the series is convergent for $|r| < 1$ and find the sum of the infinite series.

Find the sum of the infinite series

$$2 + x^{\frac{1}{3}} + \frac{1}{2}x^{\frac{2}{3}} + \frac{1}{4}x + \frac{1}{8}x^{\frac{4}{3}} + \ldots$$

and determine its radius of convergence, R. Hence find (valid for $|x| < R$) the sum of the infinite series

$$x^{-\frac{2}{3}} + x^{-\frac{1}{3}} + \frac{3}{4} + \frac{1}{2}x^{\frac{1}{3}} + \ldots$$

(12 marks)

(iii) Use l'Hôpital's rule to evaluate

$$\lim_{x\to\infty} e^{-2x}x^3 .$$

(5 marks)

2 The function $f(x) = e^x$ is defined on the range $0 \le x < 1$. Let $F(x)$ denote the Fourier *sine* series and $G(x)$ the Fourier *cosine* series of $f(x)$.

(i) Show that

$$F(x) = \sum_{n=1}^{\infty} \frac{2\pi n\,(1 - (-1)^n\,e)}{1 + (n\pi)^2} \sin n\pi x$$

(14 marks)

(ii) Sketch with two separate plots $F(x)$ and $G(x)$ for the range $-3 < x < 3$.

(7 marks)

(iii) State Fourier's Theorem and verify that it holds for $F(x)$. ***(4 marks)***

3 (i) Let $F(s)$ be the Laplace transform of $f(t)$.

(a) Find *by integration* the Laplace transform of $f(t) = 3 + t$.

(b) Using the Shift Theorem, deduce the Laplace transform of $\mathrm{e}^{-2t}(3 + t)$.

(c) Using the t-shift Theorem, find the Laplace transform of $u(t-2)\,(1+t)$, where $u(t)$ is the unit Heavyside function.

(8 marks)

(ii) With the aid of the table of Laplace transforms, find the inverse Laplace transform of

$$\frac{s-3}{(s^2-6s+34)^2}\,.$$

(3 marks)

(iii) The variables $y_1(t)$ and $y_2(t)$ satisfy the coupled system of ordinary differential equations

$$\frac{\mathrm{d}y_1}{\mathrm{d}t} = y_2\,,$$
$$\frac{\mathrm{d}y_2}{\mathrm{d}t} = -9\,y_1,$$

subject to the initial conditions $y_1(0) = 0$ and $y_2(0) = 2$. Find the Laplace transform of each equation and hence solve for $y_1(t)$. Find $y_2(t)$ without using its Laplace transform.

(6 marks)

(iv) Use the method of Laplace transforms to solve the second-order ordinary differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}\,t^2} + 6\,y = 1 - u(t-3),$$

subject to the initial conditions $y(0) = 5$ and $y'(0) = 0$.

(8 marks)

4 The temperature T on a metal disc of radius 2 units satisfies the differential equation

$$\frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = 0\,.$$

(i) Using the *method of separation of variables*, show that the general solution can be written

$$T(r,\theta) = \sum_{m=0}^{\infty} r^m \left(A_m \sin m\theta + B_m \cos m\theta\right).$$

Hint: For the radial equation consider solutions of the form r^n.

(16 marks)

(ii) If the temperature on the boundary is held at

$$T(2,\theta) = \begin{cases} 0, & -\pi \le \theta < 0, \\ 1, & 0 \le \theta < \pi, \end{cases}$$

determine the coefficients A_m and B_m. ***(9 marks)***

5 (i) Evaluate the integral

$$\int_0^1 \int_0^{1-x} (x^2 + y^2)\, \mathrm{d}y\, \mathrm{d}x\,.$$

(6 marks)

(ii) A region R is given by the bounds $x^2 + y^2 \le 1$ and $y \ge 0$. Evaluate, *using a change of coordinates*, the integral

$$\int\int_R x^2 y\ \mathrm{d}x\, \mathrm{d}y\,.$$

(9 marks)

(iii) By changing the order of integration, evaluate

$$\int_0^1 \int_{\sqrt{y}}^1 \frac{1}{\sqrt{y\,(1+x^2)}}\ \mathrm{d}x\, \mathrm{d}y\,.$$

(10 marks)

End of Question Paper

For use with MAS253 first semester examination

Formulae for use in L2 Mechanical Engineering Mathematics Examination

These results may be quoted without proof unless proofs are asked for in the question.

Trigonometry

$$\sin 2P = 2\sin P \cos P,$$

$$\cos 2P = \cos^2 P - \sin^2 P = 2\cos^2 P - 1 = 1 - 2\sin^2 P,$$

$$\cos P \cos Q = \frac{1}{2}\{\cos(P+Q) + \cos(P-Q)\},$$

$$\sin P \sin Q = -\frac{1}{2}\{\cos(P+Q) - \cos(P-Q)\},$$

$$\sin P \cos Q = \frac{1}{2}\{\sin(P+Q) + \sin(P-Q)\}.$$

Geometric progression

The partial sum to n terms of

$$a + ar + ar^2 + \ldots\ldots + ar^{n-1} + \ldots\ldots$$

is

$$S_n = a(1 - r^n)/(1 - r), \quad r \neq 1 \quad .$$

Taylor Series for functions of one variable (x)

The Taylor series of $f(x)$ about $x = a$ is

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2!} f''(a)(x-a)^2 + \ldots.$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

The Maclaurin series of $f(x)$ is the special case of the Taylor series when $a = 0$:

$$f(x) = f(0) + f'(0)\,x + \frac{1}{2!} f''(0)\,x^2 + \ldots.$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

Important examples of Maclaurin series are:

$$e^x = 1 + x + \frac{1}{2!}x^2 + \ldots \qquad (R \text{ is infinite})$$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \ldots. \qquad (R \text{ is infinite})$$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \ldots. \qquad (R \text{ is infinite})$$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \ldots \qquad (R = 1)$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha - 1)}{2!}x^2 + \ldots. \qquad (R = 1)$$

R is the radius of convergence.

Binomial Theorem

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 + \ldots..$$

If n is positive and integer, series terminates.

If n is negative or non-integer (or both), the series is an infinite series with the radius of convergence, $R = 1$.

Fourier Series

The Fourier series of $f(x)$ for $-l < x < l$ is

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right)$$

where $$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx \quad ,$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx, \quad n = 1, 2, \ldots$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx, \quad n = 1, 2, \ldots$$

Laplace Transform

The Laplace Transform of $f(t)$ is

$$F(s) = L(f(t)) = \int_0^\infty e^{-st} f(t)dt \quad .$$

For special cases, see later page.

Partial Differentiation

$$\delta F = F(x+\delta, y+\varepsilon) - F(x,y) \cong \delta \frac{\partial F}{\partial x} + \varepsilon \frac{\partial F}{\partial y}$$

Chain Rules:

1. Suppose that $F = F(x,y)$ and that x and y are functions of t, i.e. $x = x(t), y = y(t)$, then

$$\frac{dF}{dt} = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} \quad .$$

2. Suppose that $F = F(x,y)$ and that x and y are functions of the variables u and v, i.e. $x = x(u,v), y = y(u,v)$, then

$$\frac{\partial F}{\partial u} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial u}; \qquad \frac{\partial F}{\partial v} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial v} \quad .$$

Taylor Series for functions of two variables (x, y)

The Taylor series of $f(x,y)$ about $x = a, y = b$ is

$$\begin{aligned} f(x,y) &= f(a,b) + \{(x-a)\, f_x(a,b) + (y-b)\, f_y(a,b)\} + \\ &+ \frac{1}{2!}\{(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)\, f_{xy}(a,b) + \\ &+ (y-b)^2 f_{yy}(a,b)\} + \\ &+ \ldots \end{aligned}$$

Here $f_x = \dfrac{\partial f}{\partial x}$ etc.

Partial Differential Equations (2 independent variables)

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$ Laplace's equation

$$\frac{\partial^2 V}{\partial x^2} = \frac{1}{K}\frac{\partial V}{\partial t}$$ Heat conduction (or diffusion) eqn.

equation

$$\frac{\partial^2 V}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2}$$ Wave equation

General Solution of ODEs

$$X'' = -\omega^2 X \Rightarrow X(x) = A\cos \omega x + B\sin \omega x$$

$$X'' = \omega^2 X \Rightarrow X(x) = C\cosh \omega x + D\sinh \omega x$$

or $Ee^{\omega x} + Fe^{-\omega x}$

$$T' = kT \Rightarrow T(t) = Ae^{kt}$$

Table of Laplace Transforms	
$f(t)$	$F(s) = L(f(t))$
$f(t)$	$F(s)$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2 F(s) - sf(0) - f'(0)$
$f^{iv}(t)$	$s^4 F(s) - s^3 f(0) - s^2 f'(0) - sf''(0) - f'''(0)$
1	$1/s$
t	$1/s^2$
$t^{n-1}/(n-1)!\ (n \geq 1)$	$1/s^n$
e^{at}	$\dfrac{1}{s-a}$
$\dfrac{1}{a}\sin at$	$\dfrac{1}{s^2+a^2}$
$\cos at$	$\dfrac{s}{s^2+a^2}$
$\dfrac{1}{a}\sinh at$	$\dfrac{1}{s^2-a^2}$
$\cosh at$	$\dfrac{s}{s^2-a^2}$
$\dfrac{\sin at - at\cos at}{2a^3}$	$\dfrac{1}{(s^2+a^2)^2}$
$\dfrac{t\sin at}{2a}$	$\dfrac{s}{(s^2+a^2)^2}$
$e^{at} f(t)$	$F(s-a)$, where $F(s) = L(f(t))$
$\delta(t)$	1
$\delta(t-a)$	e^{-as}
$u(t-a)$	e^{-as}/s
$u(t-a)\ f(t-a)$	$e^{-as} F(s)$, where $F(s) = L(f(t))$