The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Spring Semester 2012–2013

Mathematics (Computational and Numerical Methods) — 2 hours

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) The cubic equation $x^3 - 2x - 5 = 0$ has a root near $x = 2$. Define the Newton-Raphson formula for calculating the root of a function. Starting with $x_0 = 2$, compute x_1, x_2, and x_3, the next three Newton-Raphson estimates for the root correct to three decimal places. (7 marks)

(ii) Let us consider the system of equations

$$\begin{aligned} x_1 - 0.25x_2 - 0.25x_3 &= 50 \\ -0.25x_1 + x_2 - 0.25x_4 &= 50 \\ -0.25x_1 + x_3 - 0.25x_4 &= 25 \\ -0.25x_2 - 0.25x_3 + x_4 &= 25. \end{aligned}$$

Perform three iterations using the Gauss-Seidel iterative method starting from the initial guess $[100, 100, 100, 100]^T$. Work throughout with an accuracy of three decimal places. (10 marks)

(iii) The deflection of a beam is believed to satisfy the equation

$$\frac{d^2y}{dx^2} = e^{x^2}, \quad y = y(x),$$

together with the boundary conditions $y(0) = y(1) = 0$. Estimate, using a second order finite difference approximation, the approximate deflection at 0.2, 0.4, 0.6 and 0.8. Work throughout with an accuracy of five decimal places.

Hint: The second order derivative of a function can be written in a finite difference form as

$$U''_k = \frac{U_{k+1} - 2U_k + U_{k-1}}{h^2},$$

where U_k and $U_{k\pm1}$ denote the values of $U(x_k)$ and $U(x_k \pm h)$, respectively. (8 marks)

2 (i) Find the value of the the constant C, so that one of the eigenvalues of the matrix

$$\begin{pmatrix} 2 & 4 & 3 \\ 1 & 0 & 1 \\ C & 1 & 1 \end{pmatrix}$$

is equal to -1. Further, determine the values of the remaining two eigenvalues. (10 marks)

(ii) Fit a least squares quadratic, i.e., a polynomial of degree $n = 2$, to the data $(0, 5)$, $(2, 4)$, $(4, 1)$, $(6, 6)$, $(8, 7)$. The system of equations arising should be solved using Gaussian elimination with partial pivoting. Work throughout with an accuracy of four decimal places.

Hint: Assuming that the x_i values are free of errors, the normal equations used in the process of a least squares fit for a polynomial of degree n are

$$\sum_{j=0}^{n} a_j \sum_{i=0}^{n} x_i^{j+k} = \sum_{i=0}^{n} x_i^k f_i, \quad k = 0, 1, 2, \ldots, n$$

(15 marks)

3 (i) Let

$$f(x) = e^{\sin x}$$

Show that

$$\frac{d^4 f}{dx^4} = e^{\sin x} \left(3 + \sin x - 6 \sin x \cos^2 x - 7 \cos^2 x + \cos^4 x\right)$$

and then evaluate

$$\int_0^{\pi/2} e^{\sin x} dx$$

to an accuracy of $\epsilon = 10^{-4}$ using Simpson's method assuming that $d^4 f/dx^4$ takes its maximum value at one of the ends of the interval. Work throughout with an accuracy of 4 decimal places.

Hint: If a function $f(x)$ has four continuous derivatives on an interval (a, b) and this interval is divided into n subintervals, where n is an even positive integer, then the error bound for Simpson's method is given by

$$\left|E_n^S\right| \leq \frac{h^4}{180}(b - a)K$$

where

$$h = \frac{b - a}{n}$$

and

$$K = \max_{a \leq x \leq b} \left|\frac{d^4 f(x)}{dx^4}\right|$$

(13 marks)

 Question 3 continued on next page

(ii) Factorise the matrix

$$A = \begin{pmatrix} 5 & -1 & 2 \\ -10 & 4 & 3 \\ 10 & 4 & -1 \end{pmatrix}$$

into the product $A = LU$, where L is a lower triangular matrix with unit diagonal elements and U is an upper triangular matrix.
Hence, using this factorisation, solve the matrix equation in x

$$A\mathbf{x} = \mathbf{b}$$

where $\mathbf{x} = [x_1, x_2, x_3]^T$ and $\mathbf{b} = [2, -1, 1]^T$. (12 marks)

4 (i) Maximize the objective function

$$f(x, y) = 2x + 3y$$

subject to the constraints

$$4x + 3y \geq 12, \quad x - y \geq -3, \quad y \leq 6, \quad 2x - 3y \leq 0$$

(12 marks)

(ii) A store wants to liquidate 200 of its shirts and 100 pairs of trousers from last season. They have decided to put together two offers, A and B. Offer A is a package of one shirt and a pair of trousers which will sell for £30. Offer B is a package of three shirts and a pair of trousers, which will sell for £50. The store neither wants to sell less than 20 packages of Offer A nor less than 10 of Offer B. How many packages of each do they have to sell to maximize the money generated from the promotion? Solve the problem using a geometrical approach marking clearly the feasibility region and the line of constant revenue. (13 marks)

End of Question Paper