The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2012–13**

MAS271 Methods for differential Equations **2 hours**

All FOUR questions are compulsory.

1 For the system of equations

$$\dot{x} = y(x^3 - 1), \qquad \dot{y} = 2x(y - 2).$$

(i) Find the stationary points and investigate their nature. If the stationary point is either a saddle or a node, determine the eigenvectors corresponding to particular eigenvalue. ***(15 marks)***

(ii) Find the isoclines corresponding to $\dot{x} = 0$ and $\dot{y} = 0$. Determine which isoclines consist of trajectories. ***(6 marks)***

(iii) Sketch the phase portrait of the system. ***(4 marks)***

2 (i) Determine the necessary and sufficient conditions for the function

$$f(x, y) = ax^2 + 2bxy + cy^2$$

to be positive definite. ***(6 marks)***

(ii) For the system

$$\dot{x} = -x^3 - y, \qquad \dot{y} = \mu^2 x - y^3,$$

where μ is a real parameter, find a suitable Liapunov function of the form $V(x, y) = ax^2 + 2bxy + cy^2$, where a, b and c are constants to be determined. Use this function to show that the origin is a stable stationary point. ***(6 marks)***

(iii) For what values of μ you can state the origin is asymptotically stable? Explain your answer. ***(4 marks)***

(iv) Study the stability of the origin, (0,0), for $\mu \neq 0$ using linearization. Can we state, on the basis of this analysis, that the origin is stable? State which theorem you used to reach this conclusion. ***(9 marks)***

3 (i) Find the power series solution of

$$(x-3)y' + 2y = 0.$$

Find its radius of convergence. ***(10 marks)***

(ii) Solve the following differential equations giving the general solution in each case.

$$(a)\ \ y'' + 4y' + 4y = 0$$
$$(b)\ \ y'' - 6y' + 13y = 0.$$

(6 marks)

(iii) Solve the following linear equation, subject to the given boundary condition.

$$(x+1)\frac{dy}{dx} - y = 3x^4 + 4x^3, \text{ given that } y = 1 \text{ when } x = 0.$$

(9 marks)

4 (i) Investigate the nature of the point $x = 0$ for the differential equation

$$x^4 y'' + (x^2 \sin x)y' + (1 - \cos x)y = 0.$$

(5 marks)

(ii) Find the Frobenius series solutions of

$$2x^2 y'' + 3xy' - (x^2 + 1)y = 0$$

after investigating the nature of the point $x = 0$. ***(14 marks)***

(iii) The parametric Bessel equation of order n is

$$x^2 y'' + xy' + (\alpha^2 x^2 - n^2)y = 0, \quad (*)$$

where α is a positive parameter and $'$ is differentiation with respect to x. Using $t = \alpha x$, transform $(*)$ into the standard Bessel equation

$$t^2 \frac{d^2y}{dt^2} + t\frac{dy}{dt} + (t^2 - n^2)y = 0. \quad (**)$$

(4 marks)

Give the general solution of $(*)$. ***(2 marks)***

End of Question Paper