The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2012–2013**

Probability Modelling **2 hours**

*Candidates should attempt **ALL** five questions.*
The maximum marks for the various parts of the questions are indicated.
The paper will be marked out of 100. (Q1–25; Q2–18; Q3–14; Q4–23; Q5–20)

1 Days at a particular location are dry with probability 1/3 and wet with probability 2/3, independently of each other. Define a delayed renewal process by saying that a renewal happens every time a sequence of two consecutive dry days is completed after the beginning of the year.

(a) Let v_n be the probability that a renewal occurs on day n of the year. Explain why $v_1 = 0$ and $v_n = 1/9$ for $n \geq 2$. Hence find the generating function $V(s)$, defined as $\sum_{n=0}^{\infty} v_n s^n$ for $|s| < 1$. ***(7 marks)***

(b) Let u_n be the probability that, given that a renewal occurred on day t of the year, another renewal occurs on day $t+n$. Give the values of u_0 and u_1, and the value of u_n for $n \geq 2$, giving reasons for your answers. Hence find the generating function $U(s)$, defined as $\sum_{n=0}^{\infty} u_n s^n$ for $|s| < 1$. ***(8 marks)***

(c) Using the result that, in a delayed renewal process, $V(s) = U(s)B(s)$, where $B(s)$ is the probability generating function of the time until the first renewal, find the expected number of days until the first sequence of two dry days is completed. ***(5 marks)***

(d) What is the expected number of days after the start of the year until the *second* sequence of two dry days is completed? ***(5 marks)***

2 A discrete time Markov chain has state space $S = \{1, 2, 3, 4, 5, 6\}$ and transition matrix given by

$$\begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 \end{pmatrix}.$$

(a) Find the communicating classes of the Markov chain, and state which of them are closed. ***(6 marks)***

(b) Give the period of each state of this Markov chain. Give reasons for your answers. ***(6 marks)***

(c) Assume that the chain starts in state 3. Find, giving a full reasoning for your answers, the distribution of the state of the chain at time n

(i) if $n = 3m$ for a non-negative integer m;

(ii) if $n = 3m + 1$ for a non-negative integer m;

(iii) if $n = 3m + 2$ for a non-negative integer m. ***(6 marks)***

3 Let (X_n) be a Markov chain on the state space $\{1, 2, 3\}$ with transition matrix

$$P_X = \begin{pmatrix} 1/7 & 1/7 & 5/7 \\ 1/7 & 5/7 & 1/7 \\ 2/7 & 2/7 & 3/7 \end{pmatrix}.$$

(a) Find the unique stationary distribution of the chain. ***(4 marks)***

(b) For each $j \in \{1, 2, 3\}$, give an approximate value for $P(X_n = j)$ for large n. Give a careful explanation of your reasoning. (You may use results from the course.) ***(10 marks)***

4 A coin with probability p, with $0 < p < 1$, of producing a head is tossed repeatedly, with the different tosses being independent. After toss n ($n \geq 1$) say that

$$X_n = \begin{cases} 1 & \text{toss } n \text{ was a head not immediately preceded by another head} \\ 2 & \text{toss } n \text{ was a head immediately preceded by another head} \\ -1 & \text{toss } n \text{ was a tail not immediately preceded by another tail} \\ -2 & \text{toss } n \text{ was a tail immediately preceded by another tail.} \end{cases}$$

Also say that $X_0 = 0$ with probability 1. Consider this as a Markov chain on the state space $S = \{-2, -1, 0, 1, 2\}$.

(a) Give the transition matrix, P, of the Markov chain. ***(5 marks)***

(b) Find the probability that, starting from time 0, a sequence of two heads occurs before a sequence of two tails. ***(9 marks)***

(c) Find the expected time until a sequence of two identical tosses occurs for the first time. ***(9 marks)***

5 Assume that meteor strikes on the Earth above mass m can be modelled as a Poisson process with rate $\frac{1}{5}$ per year.

(a) Find the probability that there is at least one such meteor strike in a 10-year period. ***(4 marks)***

(b) Given that there are six meteor strikes in the 20 years from 2014 to 2033 (inclusive), find the probability that exactly two of them are in the five years from 2014 to 2018. ***(5 marks)***

(c) Assuming that each such meteor strike to hit the Earth hits Russia with probability 1/20 independently of other strikes' location and timing,

(i) give the rate of the Poisson process describing meteor strikes above mass m hitting Russia; ***(2 marks)***

(ii) find the probability that at least two meteor strikes above mass m hit Russia in a given year; ***(5 marks)***

(iii) give the expected value of the time until the next meteor strike above mass m to hit Russia after the beginning of 2014. ***(4 marks)***

End of Question Paper