The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2012–2013**

Rings and Groups **2 hours**

Answer ***all four questions****.*
You should justify your answers carefully unless the question states otherwise.

1 (i) Write down all the units in the ring $\mathbb{Z}_{18}$, justifying your answer. ***(5 marks)***

(ii) What does it mean to say that a group is *cyclic*? Is the group $\mathcal{U}(\mathbb{Z}_{18})$ cyclic? Write down its class equation. ***(6 marks)***

(iii) Is it true that the units in the ring $\mathbb{Z}_{18}[x]$ are the same as those in $\mathbb{Z}_{18}$, expressed as constant polynomials? Prove this or give a counterexample. ***(4 marks)***

Additional marks for rigour and presentation. ***(5 marks)***

2 (i) Let R be a commutative ring. What does it mean for an element $r \in R$ to be *irreducible* in R? ***(2 marks)***

(ii) Let $d \neq 1$ be a square-free integer. Define $\mathcal{N}(r)$, the norm of an element $r \in \mathbb{Z}[\sqrt{d}]$. Prove that if $\mathcal{N}(r)$ is a prime number then r is irreducible in $\mathbb{Z}[\sqrt{d}]$. You may use the fact that $\mathcal{N}(st) = \mathcal{N}(s)\mathcal{N}(t)$. ***(6 marks)***

(iii) Show that the following are equivalent factorisations of 5 into irreducibles in $\mathbb{Z}[i]$:

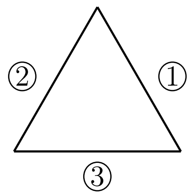
$$(1+2i)(1-2i) = 5$$
$$(2+i)(2-i) = 5.$$

Does this show that $\mathbb{Z}[i]$ is a unique factorisation domain? ***(7 marks)***

Additional marks for rigour and presentation. ***(5 marks)***

3 (i) Let $f : G \longrightarrow H$ be a group homomorphism. Define the *kernel* and *image* of f. Prove that the kernel of θ is necessarily a normal subgroup of G. You may assume it is a subgroup. Is the image of f necessarily a normal subgroup of H? Prove this or give a counter-example. ***(7 marks)***

(ii) Recall that D_3 is the group of symmetries of the equilateral triangle. Consider the equilateral triangle with edges labelled as below.



Observe that D_3 acts on the numbered edges inducing a homomomorphism $f : D_3 \longrightarrow S_3$. Write down the kernel and image of the homomorphism f. Is f injective? Is f surjective? ***(4 marks)***

(iii) State, without proof, the First Isomorphism Theorem for groups. What can you deduce about the homomorphism f in part (ii) by applying the First Isomorphism Theorem? ***(4 marks)***

Additional marks for rigour and presentation. ***(5 marks)***

4 Let G be a group of order 8, with elements a, b, c, d, e, f, g, h, and Cayley table shown below.

	a	b	c	d	e	f	g	h
a	a	b	c	d	e	f	g	h
b	b	a	d	c	f	e	h	g
c	c	d	a	b	g	h	e	f
d	d	c	b	a	h	g	f	e
e	e	f	g	h	a	b	c	d
f	f	e	h	g	b	a	d	c
g	g	h	e	f	c	d	a	b
h	h	g	f	e	d	c	b	a

(i) What is the identity element of G? Is G Abelian? Is G cyclic? ***(3 marks)***

(ii) Find a subgroup $H \subset G$, of order 2. Is H a normal subgroup? ***(3 marks)***

(iii) Write down all the left cosets xH. ***(4 marks)***

(iv) Recall that the left cosets are the elements of the quotient group G/H. What is the order of G/H in this example? Write out the Cayley table for the group G/H. Is G/H isomorphic to the cyclic group of order 4 or the Klein 4-group? ***(5 marks)***

Additional marks for rigour and presentation. ***(5 marks)***

End of Question Paper