The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2012–2013**

Nonlinear Mathematics **2 hours**

*Answer **four** questions. You are advised **not** to answer more than four questions: if you do, only your best four will be counted.*

Throughout the paper $\mathrm{I}(a,b)$ *denotes the open interval* $\{t \in \mathbb{R} \mid a < t < b\}$ *and* I *denotes an open interval in* $\mathbb{R}$ *with unspecified endpoints.*

1 (i) Let $\varphi\colon\ M \to N$ be a map where $M \subseteq \mathbb{R}^4$ and $N \subseteq \mathbb{R}^3$.

(a) For $(u, v, w) \in N$ define the preimage $\varphi^{-1}(u, v, w)$. **(2 marks)**

Now define $\varphi\colon\ \mathbb{R}^4 \to \mathbb{R}^3$ by

$$\varphi(x, y, z, t) = (2(xt - yz),\ 2(xy + zt),\ x^2 - y^2 + z^2 - t^2).$$

Find the following preimages:

(b) $\varphi^{-1}(1, 0, 0)$; **(10 marks)**

(c) $\varphi^{-1}(0, 0, 0)$. **(8 marks)**

(ii) Let $F\colon\ \mathbb{R}^2 \to \mathbb{R}$ be a function and define $\varphi\colon\ \mathbb{R}^2 \to \mathbb{R}^2$ by $\varphi(x, y) = (F(x, y), y)$.

Assume that φ is a bijection.

Show that there is a map $\alpha\colon\ \mathbb{R} \to \mathbb{R}$ such that $F(\alpha(y), y) = 0$ for all $y \in \mathbb{R}$. **(5 marks)**

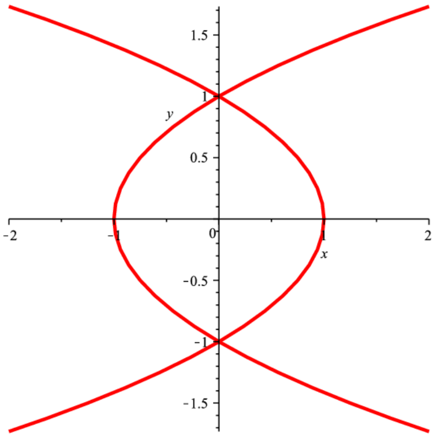

Figure 1: For Question 2(d).

2 (a) Let $(a, b) \in \mathbb{R}^2$ and let $r > 0$. Define the open ball $B((a, b), r)$.

Define what it means for a set $M \subseteq \mathbb{R}^2$ to be open. **(5 marks)**

(b) Let $M_1 \subseteq \mathbb{R}^2$ and $M_2 \subseteq \mathbb{R}^2$ be any open sets. Prove that the intersection $M_1 \cap M_2$ is open. **(4 marks)**

(c) For a function $F\colon M \to \mathbb{R}$ where $M \subseteq \mathbb{R}^2$ is a set, and $I \subseteq \mathbb{R}$ is an interval, define the set $F^{-1}(I)$.

State carefully (but do not prove) the theorem from lectures which ensures that, under certain conditions, $F^{-1}(I)$ is an open set in $\mathbb{R}^2$. **(6 marks)**

(d) In Figure 1, the two parabolas have equations $y^2 = 1 + x$ and $y^2 = 1 - x$.

Denote the region between the two parabolas, not including the parabolas themselves, by M. Using (b) and (c) or otherwise, prove that M is an open set. **(10 marks)**

3 (i) (a) Define what it means for a function $F\colon M \to \mathbb{R}$, where $M \subseteq \mathbb{R}^2$ is an open set, to be continuous at $(a,b) \in M$.

(b) Define what it means for a function $F\colon M \to \mathbb{R}$, where $M \subseteq \mathbb{R}^2$ is an open set, to be C^1.

(c) Prove that, for all $x, y \in \mathbb{R}$,

$$2|xy| \leqslant x^2 + y^2.$$

(10 marks)

(ii) Define $F\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$F(x,y) = \begin{cases} \dfrac{x^2 y}{x^2+y^2} & (x,y) \neq (0,0), \\ 0 & (x,y) = (0,0). \end{cases}$$

(a) Prove that F is continuous on $\mathbb{R}^2$. **(5 marks)**

(b) Find $\dfrac{\partial F}{\partial x}(x,y)$ for all $(x,y) \in \mathbb{R}^2$. **(4 marks)**

(c) Show that $\dfrac{\partial F}{\partial x}$ is not continuous at $(0,0)$. **(6 marks)**

4 (a) Let M and N be open subsets of $\mathbb{R}^3$. Define what it means for a map $\varphi\colon M \to N$ to be a diffeomorphism.

(3 marks)

(b) State carefully and in full the Local Diffeomorphism Theorem for maps $M \to N$ where M and N are open sets in $\mathbb{R}^3$. **(6 marks)**

(c) Now consider the map $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^3$, $\varphi(x,y,z) = (yz, zx, xy)$.

- Find $D(\varphi)(x,y,z)$ and $\det D(\varphi)(x,y,z)$.
- Write M for the set $\{(x,y,z) \in \mathbb{R}^3 \mid \det D(\varphi)(x,y,z) = 0\}$ and write

  $$S = \{(u,v,w) \in \mathbb{R}^3 \mid uvw = 0\}.$$

  Show that $(x,y,z) \in M$ if and only if $\varphi(x,y,z) \in S$.
- Let $M' = \{(x,y,z) \in \mathbb{R}^3 \mid x > 0,\ y > 0,\ z > 0\}$ and let $N' = \{(u,v,w) \in \mathbb{R}^3 \mid u > 0,\ v > 0,\ w > 0\}$.

  Show that if $(x,y,z) \in M'$ then $\varphi(x,y,z) \in N'$.
- Show that the restriction of φ to $M' \to N'$ is a diffeomorphism, giving an explicit formla for the inverse $\varphi^{-1}\colon N' \to M'$.

(16 marks)

5 (i) State carefully and in full the version of the Implicit Function Theorem which applies to maps with domain an open set $M \subseteq \mathbb{R}^3$ and codomain $\mathbb{R}^2$ and which gives conditions for a solution in terms of z. **(10 marks)**

(ii) Consider the simultaneous equations

$$(x^2 + z^2)^2 = x^2 + 3y^2, \qquad x^2 + y^2 = 8. \tag{$*$}$$

The point $(2, 2, 0)$ is a solution.

(a) Determine whether the hypotheses of the Implicit Function Theorem, as you stated it in (i), are satisfied at $(2, 2, 0)$. **(5 marks)**

(b) Solve the equations explicitly for x and y in terms of z. If you found that the hypotheses of the Implicit Function Theorem are satisfied, give your solution in terms of functions $\alpha_1 \colon I \to \mathbb{R}$ and $\alpha_2 \colon I \to \mathbb{R}$ for an open interval $I \subseteq \mathbb{R}$, and determine a suitable I explicitly. If you found the hypotheses to not be satisfied, determine whether your solution is isolated, or fails to be unique, or fails to be C^1. **(10 marks)**

End of Question Paper