The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Autumn Semester 2012–2013

Continuum Mechanics — 2 hours

*Answer **four** questions. You are advised **not** to answer more than four questions: if you do, only your best four will be counted.*

1 (i) The new Cartesian coordinates x_1', x_2', x_3' are obtained from the old ones, x_1, x_2, x_3, by rotating the coordinate axes about the x_2-axis by the angle θ. You are given that the matrix of transformation from the old to the new coordinates is

$$\hat{\mathbf{A}} = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}.$$

Tensor $\mathbf{T}$ is invariant with respect to the rotation about the x_2-axis. Show that, in Cartesian coordinates x_1, x_2, x_3, its matrix, $\hat{\mathbf{T}}$, has the form

$$\hat{\mathbf{T}} = \begin{pmatrix} T_1 & 0 & T_3 \\ 0 & T_2 & 0 \\ -T_3 & 0 & T_1 \end{pmatrix}.$$

(You can use without proof that the matrices of components of the tensor $\mathbf{T}$ in the old and new coordinates are related by $\hat{\mathbf{T}}' = \hat{\mathbf{A}}\hat{\mathbf{T}}\hat{\mathbf{A}}^T$.) ***(12 marks)***

(ii) The trace of matrix $\hat{\mathbf{T}}$ is determined by $\mathrm{tr}(\hat{\mathbf{T}}) = T_{ii}$. Calculate $\mathrm{tr}(\hat{\mathbf{T}})$. Show that

$$\mathrm{tr}\big(\hat{\mathbf{T}}^2\big) = 2T_1^2 + T_2^2 - 2T_3^2.$$

(4 marks)

(iii) The first, second and third invariants of matrix $\hat{\mathbf{T}}$ are determined by $I_1 = \mathrm{tr}(\hat{\mathbf{T}})$, $I_2 = \frac{1}{2}\big[I_1^2 - \mathrm{tr}\big(\hat{\mathbf{T}}^2\big)\big]$ and $I_3 = \det(\hat{\mathbf{T}})$. You are given that $I_1 = I_2 = 0$, $I_3 = 27$, and $T_3 > 0$. Find T_1, T_2 and T_3. ***(9 marks)***

2 A body is subjected to the deformation

$$x_1 = \xi_1, \qquad x_2 = (1+\beta)\xi_2, \quad x_3 = \kappa\xi_2 + (1+\beta)\xi_3,$$

where $\beta > 0$ and $\kappa \geq 0$.

(i) Interpret the deformation geometrically when $\kappa = 0$. Draw the sketch. ***(5 marks)***

(ii) Show that there is an increase in volume. ***(5 marks)***

(iii) In the initial configuration, a line element of length dl_0 lies in the plane $\xi_1 = 0$ and is inclined at an acute angle θ to the ξ_2 axis so that

$$d\xi_1 = 0, \quad d\xi_2 = dl_0 \cos\theta, \quad d\xi_3 = dl_0 \sin\theta.$$

Show that the length dl of this element after deformation is given by

$$dl = \left\{(1+\beta)^2 + 2\kappa(1+\beta)\sin\theta\cos\theta + \kappa^2\cos^2\theta\right\}^{1/2} dl_0.$$

Hence, show that all such elements are stretched during the deformation. Calculate the angle between the element $\mathbf{x}$ and the ξ_2-axis. Is it smaller or larger than θ? ***(15 marks)***

3 (i) Write down the expression for the surface traction, $\mathbf{t}$, in terms of the stress tensor, $\mathbf{T}$, and the unit normal to the surface, $\mathbf{n}$. Express it both in vector and coordinate form. ***(3 marks)***

(ii) By considering the equilibrium of an infinitesimal cube with sides parallel to the x_1, x_2, x_3 coordinate axes, derive the equilibrium equation in Cartesian coordinates

$$\frac{\partial T_{ij}}{\partial x_j} + \rho b_i = 0,$$

where T_{ij} are the components of the stress tensor $\mathbf{T}$, ρ is the density, and b_i are the components of the body force $\mathbf{b}$. Draw a sketch clearly indicating the forces acting on the cube. ***(11 marks)***

(iii) A charged medium with the density $\rho = \text{const}$ and the electric charge density $\rho_e = \text{const}$ is in equilibrium in a gravity field with the gravity acceleration $\mathbf{g} = \text{const}$, and the electric field $\mathbf{E} = \text{const}$. You are given that $\mathbf{g}$ is antiparallel to the x_3-axis and $\mathbf{E}$ is parallel to the x_1-axis. The body force imposed on the medium by the electric field is equal to $(\rho_e/\rho)\mathbf{E}$. You are also given that the matrix of the stress tensor has the form

$$\hat{\mathbf{T}} = \begin{pmatrix} T_1 & 0 & T_3 \\ 0 & T_1 & 0 \\ T_3 & 0 & T_2 \end{pmatrix},$$

$\mathbf{T} = 0$ at $x_3 = 0$, and $T_1 = h x_1^2 x_3$, where h is a given constant. Calculate T_2 and T_3. ***(11 marks)***

4 (i) The motion of an ideal fluid is called potential if the velocity can be written in the form $\mathbf{v} = \nabla\varphi$, φ being called the velocity potential. By using Euler's equation for incompressible homogeneous fluid written in the Gromeka-Lamb form,

$$\frac{\partial \mathbf{v}}{\partial t} + (\nabla \times \mathbf{v}) \times \mathbf{v} = -\nabla\left(\frac{p}{\rho} + \frac{1}{2}\|\mathbf{v}\|^2 + \Pi\right),$$

where Π is the body force potential, derive the Lagrange-Cauchy integral for fluid potential motion,

$$\frac{\partial \varphi}{\partial t} + \frac{p}{\rho} + \frac{1}{2}\|\mathbf{v}\|^2 + \Pi = f(t),$$

where $f(t)$ is an arbitrary function of time. ***(5 marks)***

(ii) A tank in the form of a cylinder has radius $R = 1$ m and height $H = 2$ m. The tank is open from above and filled with water up to the top. There is a circular hole of radius $r = 1$ cm at the bottom of the tank, so that water leaks through this hole.

(a) Use the Lagrange-Cauchy integral to show that

$$v^2 = 2gh,$$

where $v = \|\mathbf{v}\|$ is the water speed at the hole, and h is the water depth in the tank. You can assume that the water is an ideal incompressible fluid, and the water motion is potential and quasi-stationary, so that you can neglect the term $\partial\varphi/\partial t$ and take $f(t)$ to be constant in the Lagrange-Cauchy integral. ***(5 marks)***

(b) Show that h satisfies the equation

$$\frac{dh}{dt} = -\frac{r^2\sqrt{2gh}}{R^2}.$$

(5 marks)

(c) Using the numerical values given in the problem, determine the period of time after which only a half of the initial volume of water remains in the tank. (You can take $g \approx 10 \text{ m s}^{-2}$.) ***(10 marks)***

5 (i) You are given that, in equilibrium, the stress tensor $\mathbf{T}$ satisfies the equation written in Cartesian coordinates x_1, x_2, x_3,

$$\frac{\partial T_{ij}}{\partial x_j} + \rho b_i = 0, \qquad (*)$$

where T_{ij} are the components of the stress tensor $\mathbf{T}$, ρ is the density, and b_i are the components of the body force $\mathbf{b}$. You are also given that, in linear elasticity, the Cartesian components of the stress tensor are given by

$$T_{ij} = \lambda \delta_{ij} \frac{\partial u_k}{\partial x_k} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right),$$

where u_i are the Cartesian components of the displacement $\mathbf{u}$, and λ and μ are the Lamé constants. Show that, in the linear elasticity, equation $(*)$ reduces to

$$(\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) + \mu \nabla^2 \mathbf{u} + \rho \mathbf{b} = 0. \qquad (\dagger)$$

(5 marks)

(ii) There is an infinite elastic tube of internal radius a and external radius b. Initially the pressure inside and outside the tube is equal to the atmospheric pressure p_a. Then the pressure inside the tube is increased to $p_0 > p_a$.

(a) Introduce cylindrical coordinates r, φ, z with the z-axis coinciding with the tube axis. Neglecting the body force ($\mathbf{b} = 0$), and assuming that the displacement of the tube material is in the radial direction ($\mathbf{u} = u\mathbf{e}_r$, where $\mathbf{e}_r$ is the unit vector in the radial direction), and depends on r only ($u = u(r)$), show that equation $(\dagger)$ reduces to

$$\frac{d}{dr}\left(\frac{1}{r} \frac{d(ru)}{dr} \right) = 0.$$

$\left[\text{you can use without proof that } \nabla^2(u\mathbf{e}_r) = \mathbf{e}_r \frac{d}{dr}\left(\frac{1}{r}\frac{d(ru)}{dr} \right) \text{ and } \nabla \cdot (u\mathbf{e}_r) = \frac{1}{r}\frac{d(ru)}{dr}\right]$. Then show that the displacement is given by

$$u = Ar + \frac{B}{r},$$

where A and B are constants. ***(5 marks)***

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5 (continued)

(b) You are given that the surface traction at the tube boundaries is given by

$$\mathbf{t}(r_0) = \left(\frac{\lambda}{r} \frac{d(ru)}{dr}\bigg|_{r=r_0} + 2\mu \frac{du}{dr}\bigg|_{r=r_0} \right) \mathbf{n},$$

where $r_0 = a$ and $\mathbf{n} = -\mathbf{e}_r$ at the internal boundary, and $r_0 = b$ and $\mathbf{n} = \mathbf{e}_r$ at the external boundary. Use the boundary conditions, $\mathbf{t} = p_0\mathbf{e}_r$ at $r = a$ and $\mathbf{t} = -p_a\mathbf{e}_r$ at $r = b$, to determine the increase in the external radius of the tube δ. Calculate the numerical value of δ if $a = 9$ cm, $b = 10$ cm, $p_a = 10^5$ N/m^2, $p_0 = 10p_a$, and $\lambda = \mu = 10^8$ N/m^2. ***(15 marks)***

End of Question Paper