Data provided:
Graph Paper, Laplace Transform Table

MAS312

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Autumn Semester 2012–13

Classical Control Theory — 2 hours

*Answer **four** questions. You are advised **not** to answer more than four questions: if you do, only your best four will be counted.*

1 (i) A Galitzin seismograph has dynamic behaviour modelled by

$$\ddot{x}(t) + 2K_1\dot{x}(t) + n_1^2 x = \lambda\ddot{u}(t)$$

$$\ddot{y}(t) + 2K_2\dot{y}(t) + n_2^2 y(t) = \mu\dot{x}(t)$$

where $y(t)$ is the displacement of the mirror, $x(t)$ is the displacement of the pendulum and u is the ground displacement.

$K_1 = K_2 = n_1 = n_2 = n > 0$, and $\lambda > 0$, $\mu > 0$ are constants.

Determine the response $y(t)$ for $t > 0$ to a unit ground velocity shift $\dot{u}(t) = h(t)$, the Heaviside function, assuming zero initial conditions

$$x(0) = \dot{x}(0) = y(0) = \dot{y}(0) = u(0) = \dot{u}(0) = 0$$

(8 marks)

(ii) Draw the root-locus diagram of the the constant-gain feedback system where the open-loop transfer function is

$$G(s) = \frac{s+6}{(s+3)(s^2+2s+5)}$$

with constant-gain K $(K \geq 0)$. Find the centre of the asymptotes and the angles of the asymptotes, and the angles of departure from the complex pole pair. Also find the value of the gain K at crossover to the right-half plane, and the pure-imaginary closed-loop poles for this value of K.

(17 marks)

2 (i) A function $z(t)$ is a smooth continuous and differentiable function. Prove that the Laplace transform of the derivative of the function $z(t)$ is

$$\mathcal{L}\left(\frac{dz}{dt}\right) = sZ(s) - z(0)$$

(4 marks)

(ii) A linear system is described by the system of linear differential equations

$$\begin{aligned} \frac{dx(t)}{dt} + 3x(t) + 2y(t) &= 0 \\ \frac{dy(t)}{dt} - 2x(t) + 3y(t) &= u(t)\ . \end{aligned} \qquad (1)$$

Take the Laplace transform of these equations, assuming zero initial conditions. Eliminate the variable $X(s)$ to find the transfer function $G(s)$ relating the input u to the output y in the s-domain. ***(5 marks)***

Find the poles and zeros of this system and assess its stability. Find the impulse response $g(t) = \mathcal{L}^{-1}(G(s))$ for $t \geq 0$.

Using the partial fractions expansion of $Y(s)$, find the output $y(t)$ for $t \geq 0$ when the input is the unit Heaviside step function $u(t) = h(t)$ and the initial conditions are zero. ***(8 marks)***

Using the partial fractions expansion of $Y(s)$, find the output $y(t)$ for $t \geq 0$ when the input is $u(t) = e^{-2t}h(t)$ and the initial conditions are zero.

(8 marks)

3 (i) The input $u(t)$ to a linear system with transfer function

$$G(s) = \frac{1}{(s+1)(s^2+2s+2)}$$

is the sinusoidal function $u(t) = 5\cos 3t$ and the output is $y(t)$. Using a Laplace Transform approach determine the output $y(t)$.

Verify that the amplitude of the steady state output sinusoid equals $5|G(3i)|$. ***(12 marks)***

(ii) Use the Routh-Hurwitz criterion to determine the stability of each of the polynomials below:

(a) $s^4 + 2s^3 + 3s^2 + 4s + 5$

(b) $s^4 + s^3 - 2s^2 - 3s - 3$

(5 marks)

 Question 3 continued on next page

3 (continued)

(iii) The characteristic equation of a feedback system is

$$s^3 + 3(k+1)s^2 + (7k+5)s + (4k+9) = 0$$

By applying the Routh-Hurwitz Criterion, determine the positive values of k such that the roots of the above equation lie to the left of the line $s = -1$ in the complex plane. ***(8 marks)***

4 (i) The open-loop system with transfer function

$$G(s) = \frac{s^2 + 4}{(s^2 + 3s + 2)(s - p)}$$

is unstable for $p > 0$. This problem will show that, *using constant-gain feedback, it is possible to obtain a stable closed-loop system only for certain values of the open-loop pole p and the gain.*

Consider $p = 0.5$.

(a) Set up the Routh-Hurwitz Table to determine the closed-loop stability of the constant-gain feedback system and show that the closed-loop system can be made stable for gain k in a certain interval. Determine this interval. ***(7 marks)***

(b) Draw the root-locus diagram of the above system. Make sure you compute the relevant angles of arrival at the complex zero pair. You can assume that the break point on the real axis is approximately at the mid-point of the relevant interval. ***(10 marks)***

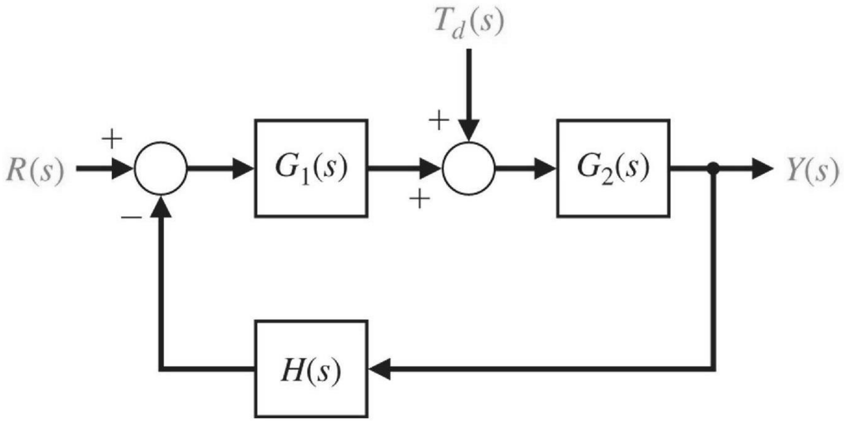
 Question 4 continued on next page

4 (continued)

(ii) A self-propelled underwater vehicle (depicted in the figure below) has the transfer function

$$G_2(s) = \frac{1}{Js^2}\,, \quad J > 0$$

We desire to control the output $Y(s)$ to track a reference input $R(s)$. The system has a negative unity feedback control structure with $H(s) = 1$. The controller $G_1(s)$ is in the forward path. The additive underwater current disturbance is $T_d(s)$.



For $G_1(s) = K_p + K_d s$ determine the range of values K_p and K_d for the system to be stable. Determine the steady state error to a unit step disturbance assuming $R(s) = 0$. ***(8 marks)***

5 (i) State the Nyquist stability criterion for a negative feedback closed-loop system with positive gain value K that may have unstable open-loop poles.

With reference to a Nyquist plot, explain the definition of the **gain margin**.

(5 marks)

(ii) Sketch an approximation of the Nyquist plot of the constant-gain feedback system with open-loop transfer function

$$G(s) = \frac{5(s+10)}{(s+1)(s+2)(s+3)}.$$

Hence, calculate the range of positive gain values for which the system is stable. ***(13 marks)***

(iii) A constant-gain feedback system has the OL transfer function

$$G(s) = \frac{K}{s+1}$$

and a time delay of T time units in the forward loop. The time delay has the transfer function $G_D(s) = \exp(-sT)$ with $T = 1$. Sketch an approximation of the Nyquist plot.

Verify that the first intercept (as ω increases) on the negative real axis has $\omega \approx 2.0288$ rad/sec. Determine the positive value K_s such that the system is stable for $0 < K < K_s$.

(7 marks)

Table of Laplace Transform Pairs

Time Function	Laplace Transform
$h(t)$	$\frac{1}{s}$
$\frac{t^n}{n!}$	$\frac{1}{s^{n+1}}$
e^{-at}	$\frac{1}{s+a}$
$\frac{t^n e^{-at}}{n!}$	$\frac{1}{(s+a)^{n+1}}$
$\cos\omega_0 t$	$\frac{s}{s^2+\omega_0^2}$
$\sin\omega_0 t$	$\frac{\omega_0}{s^2+\omega_0^2}$
$e^{-at}\cos\omega_0 t$	$\frac{s+a}{(s+a)^2+\omega_0^2}$
$e^{-at}\sin\omega_0 t$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$

End of Question Paper