The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2012–13**

INTRODUCTION TO RELATIVITY

2 hours

*Marks will be awarded for your best **FOUR** answers.*

1 (i) Define what is meant by an *inertial frame* and by an *event*. ***(4 marks)***

(ii) Two inertial frames, $R:\ (ct, x)$ and $\tilde{R}:\ (c\tilde{t}, \tilde{x})$ are related by the Lorentz transformation

$$\begin{pmatrix} c\tilde{t} \\ \tilde{x} \end{pmatrix} = \gamma(u) \begin{pmatrix} 1 & \frac{u}{c} \\ \frac{u}{c} & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix},$$

where

$$\gamma(u) = \left(1 - \frac{u^2}{c^2}\right)^{-\frac{1}{2}}.$$

An event E at the origin O of the inertial frame R has co-ordinates $(ct, 0)$. Find the co-ordinates of the event E in the frame $\tilde{R}$. Hence show that the origin of R moves with speed u relative to the inertial frame $\tilde{R}$.

(5 marks)

(iii) An event $F = (ct, ct)$ lies on a light ray travelling with speed c along the x-axis of the inertial frame R.

Find the co-ordinates of the event F in the inertial frame $\tilde{R}$ and hence show that $g(F, F)$ is invariant under the Lorentz transformation. ***(3 marks)***

(iv) Let $A = (cT_A, X_A)$ and $B = (cT_B, X_B)$ be two events in the inertial frame R.

Find the co-ordinates of the events A and B in the inertial frame $\tilde{R}$. And, show that the two events occur at the same position in $\tilde{R}$ if

$$X_A - X_B = -u\,(T_A - T_B)\,.$$

(6 marks)

(v) Show that the two events are simultaneous in the frame $\tilde{R}$ if

$$T_A - T_B = -\frac{u}{c^2}\,(X_A - X_B)\,.$$

Show that the temporal *order* of the two events A and B is the same in both inertial frames if

$$\frac{X_B - X_A}{c\,(T_A - T_B)} < \frac{c}{u}.$$

(7 marks)

2 (i) Two inertial frames $R : (ct, x, y, z)$ and $\tilde{R} : (c\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z})$ are related by the transformation

$$\begin{pmatrix} c\tilde{t} \\ \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = L \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}, \qquad (*)$$

where L is a (4×4)-matrix with constant entries.

Consider the transformation $(*)$ with the matrix L given by

$$L = \frac{1}{\sqrt{2}} \begin{pmatrix} -2 & 1 & 0 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

Show that the transformation $(*)$ is a Lorentz transformation in this case, and determine whether the transformation is

(a) proper,

(b) orthochronous.

(20 marks)

(ii) A particle has the four-velocity V and four-acceleration A in an inertial frame R. By using $g(A, V) = 0$, show that

$$g(A, A) + g\left(V, \frac{dA}{d\tau}\right) = 0,$$

where τ is a proper time. ***(5 marks)***

3 (i) Two inertial frames, $R : (ct, x, y, z)$ and $\tilde{R} : (c\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z})$ are related by the Lorentz transformation

$$\begin{pmatrix} c\tilde{t} \\ \tilde{x} \end{pmatrix} = \gamma(v) \begin{pmatrix} 1 & \frac{v}{c} \\ \frac{v}{c} & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix},$$

where

$$\gamma(v) = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}},$$

and

$$\tilde{y} = y, \qquad \tilde{z} = z.$$

A four-vector X has components $(\alpha, \beta, 0, 0)$ in the frame R, and you are given that $\beta > 0$.

The four-vector X is *null* and *past-pointing.*

Find the constant α in terms of β. ***(4 marks)***

(ii) Using this expression for α in terms of β:

(a) Find the components of X in the frame $\tilde{R}$.

(b) Show that X is past-pointing in the inertial frame $\tilde{R}$.

(c) Find the value of v such that the components of X in the frame $\tilde{R}$ are $(-1, 1, 0, 0)$. ***(10 marks)***

(iii) The four-vector $Y = (a, b, 0, 0)$ (where $b > 0$) is time-like and future-pointing.

Working in the inertial frame in which the components of X are $(-1, 1, 0, 0)$, give possible values of a and b if the four-vector $Z = X + Y$ is (a) time-like and future-pointing, (b) null and future-pointing, and (c) space-like.

(11 marks)

4 A particle moves along the x-axis of an inertial frame R such that its position x at time t in the frame is given by

$$x = \frac{1}{2}at^2,$$

where a is a constant.

(i) Find the velocity and acceleration of the particle as measured in the frame R. ***(3 marks)***

(ii) Show that the proper time τ of the particle is given in terms of the co-ordinate time t by

$$\frac{d\tau}{dt} = \left(1 - \frac{a^2t^2}{c^2}\right)^{\frac{1}{2}}.$$

(4 marks)

(iii) Hence find the four-velocity of the particle in the frame R. ***(4 marks)***

(iv) Find the four-acceleration of the particle in the frame R. ***(10 marks)***

(v) Hence find the acceleration experienced by the particle in its own rest frame. ***(4 marks)***

5 (i) Define the *four-momentum* of a particle of rest mass m and four-velocity U. ***(1 mark)***

(ii) Two particles have rest masses m_1 and m_2 and four-velocities U_1 and U_2 respectively.

The *relativistic centre of mass* of the two particles is defined to have four-velocity W, where

$$m_1 U_1 + m_2 U_2 = MW,$$

for some $M > 0$.

Explain why the four-velocity of the relativistic centre-of-mass of the two particles is unchanged when the particles collide. ***(2 marks)***

(iii) A bullet of rest-mass m_1 is travelling with speed u along the x-axis of an inertial frame R when it strikes a stationary target of rest mass m_2.

Show that, in R, the speed of the relativistic centre-of-mass is

$$w = \frac{m_1 \gamma(u) u}{m_1 \gamma(u) + m_2},$$

where

$$\gamma(u) = \left(1 - \frac{u^2}{c^2}\right)^{-\frac{1}{2}},$$

and that

$$M^2 = m_1^2 + m_2^2 + 2 m_1 m_2 \gamma(u).$$

(10 marks)

(iv) In the inertial frame in which the centre-of-mass is at rest, suppose that the bullet has speed u_1 and the target speed u_2.

Show that

$$\gamma(u_1) = \frac{M^2 + m_1^2 - m_2^2}{2 M m_1}$$

and derive the corresponding expression for $\gamma(u_2)$ in terms of m_1, m_2 and M. ***(8 marks)***

(v) Now suppose that the bullet collides *elastically* with the target, so that neither changes rest-mass during the collision. Show that, in the inertial frame in which the centre-of-mass is at rest, the speed of neither the bullet nor the target changes in the collision. ***(4 marks)***

End of Question Paper