The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Autumn Semester 2012–13

WAVES — 2 hours

Marks will be awarded for your best FOUR *answers. The marks awarded to each question or section of question are shown in italics.*

1 A string with uniform mass per unit length ρ is under a uniform tension F, and undergoes small transversal motion. The displacement of the point distance x along the string at time t is $y(x, t)$. You are given that, to adequate approximation,

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}, \tag{1}$$

where the positive constant c is defined by $c^2 = F/\rho$.

(i) By changing to new dependent variables u and v, where $u = x - ct$ and $v = x + ct$, find the general solution of (1).

(10 marks)

Suppose that the string is infinite in both directions and that

$$y(x, 0) = 0 \ \ (-\infty < x < \infty); \quad y_t(x, 0) = \Psi(x) \ \ (-\infty < x < \infty).$$

You are *given* that, for all $t \geq 0$, it can be shown that

$$y(x, t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \Psi(s)\, ds.$$

Consider the special case when

$$\Psi(x) = \begin{cases} 0, & x < -a, \\ V, & |x| < a, \\ 0, & x > a, \end{cases}$$

where V and a are positive constants.

(ii) When $ct = \frac{1}{2}a$, show that $y(x, t) = \frac{Va}{2c}$ for $|x| < \frac{1}{2}a$.

(3 marks)

1 (continued)

(iii) When $ct = \frac{1}{2}a$, find $y(x,t)$ for *(a)* $|x| > \frac{3}{2}a$, *(b)* $-\frac{3}{2}a < x < -\frac{1}{2}a$, *(c)* $\frac{1}{2}a < x < \frac{3}{2}a$.

(5 marks)

(iv) Sketch the graph of y against x when $ct = \frac{1}{2}a$.

(3 marks)

(v) Show that $y(0,t)$ has the same value for all $ct > a$, and state this value.

(4 marks)

2 A uniform string of finite length l and uniform density ρ undergoes small transverse vibrations with displacement $y(x,t)$, where $c^2 y_{xx} = y_{tt}$, and c^2 is a constant. The tension F in the string is equal to ρc^2. You are given that

(a) $y(0,t) = y(l,t) = 0$;

(b) $y(x,0) = 0$;

(c) $\dot{y}(x,0) = (V/l^2)x(l-x)$ where V is a constant;

(d) $y(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{n\pi ct}{l}\right)$, where the b_n $(n = 1, 2, 3, ..)$ are constants.

You may *assume* that the series in (d) satisfies $c^2 y_{xx} = y_{tt}$ and conditions (a) and (b).

(i) Find the b_n so that condition (c) is satisfied.

(9 marks)

(ii) Deduce that the total (kinetic plus potential) energy E is constant and given by

$$E = \frac{16V^2\rho l}{\pi^6}\sum_{p=0}^{\infty}\frac{1}{(2p+1)^6}.$$

[*Here you may assume that E is the sum of the energies in each of the normal modes.*]

(11 marks)

(iii) By equating E to the initial kinetic energy of the string, deduce that

$$\sum_{p=0}^{\infty}\frac{1}{(2p+1)^6} = \frac{\pi^6}{960}.$$

(5 marks)

3 In a compressible, static and uniform gas, with constant density ρ_0 and pressure p_0, due to the passage of a sound disturbance, there are *small* changes in density ρ and pressure p.

(i) Given that the exact equation of continuity is

$$\rho_t + (\rho u)_x + (\rho v)_y + (\rho w)_z = 0,$$

where $\mathbf{u}(\mathbf{x}, t) = u(\mathbf{x}, t)\mathbf{i} + v(\mathbf{x}, t)\mathbf{j} + w(\mathbf{x}, t)\mathbf{k}$ is the velocity field of the perturbed state, obtain a valid approximation to the exact continuity equation in the limit of linear theory (i.e. small changes).

(5 marks)

(ii) Using Newton's Second Law, again in linear approximation, and given that p is a function of ρ, show that

$$\rho_{tt} = c^2(\rho_{xx} + \rho_{yy} + \rho_{zz}),$$

where c^2 is a constant which should be defined.

(10 marks)

(iii) In a particular case $\rho = \rho_0(1 + s)$ and supposing a potential flow (i.e. $u = \phi_x$, $v = \phi_y$ and $w = \phi_z$, where ϕ is the velocity potential) show that for the linearised continuity equation

$$s_t = -(\phi_{xx} + \phi_{yy} + \phi_{zz}).$$

Provided that conditions are steady as $|\ \mathbf{x}\ | \to \infty$, deduce that

$$c^2 s = -\phi_t,$$

and hence show that

$$\phi_{tt} = c^2(\phi_{xx} + \phi_{yy} + \phi_{zz}),$$

i.e. the velocity potential also satisfies the three-dimensional form of the wave equation.

(10 marks)

4 The equilibrium position of the free surface of a liquid of depth h is $z = 0$, where z is measured vertically upwards. A surface wave causes the displacement of this surface to be $\eta(x, t)$, where x is measured along the free surface and

$$\eta = a \sin kx \cos \omega t,$$

where a, k and ω are positive constants with $ka \ll 1$. You are given that the velocity potential $\phi = \phi(x, z, t)$ satisfies

$$\text{(a)}\ \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} = 0; \qquad \text{(b)}\ \frac{\partial\phi}{\partial z} = 0 \ \text{ at }\ z = -h;$$

$$\text{(c)}\ \frac{\partial\phi}{\partial z} = \frac{\partial\eta}{\partial t} \ \text{ at }\ z = 0; \quad \text{(d)}\ \frac{\partial\phi}{\partial t} + g\eta = 0 \ \text{ at }\ z = 0.$$

(i) Explain briefly why each of (a), (b), (c) and (d) hold.

(8 marks)

(ii) Show that all conditions can be satisfied by taking

$$\phi = f(z) \sin kx \sin \omega t$$

for a suitable $f(z)$, which is to be found, and provided

$$\omega^2 = gk \tanh(kh).$$

(11 marks)

(iii) Determine the phase velocity c and the group velocity c_g in terms of g, k and h. Show that

$$\frac{c_g}{c} = \frac{1}{2}\left[1 + \frac{2kh}{\sinh 2kh}\right].$$

(6 marks)

5 In a model of traffic flow in the direction of Ox, the density of traffic at time t is $\rho(x, t)$, the speed of traffic of density ρ is $v = v(\rho)$, the flowrate $q(\rho) = \rho v(\rho)$, and $c(\rho) = q'(\rho)$.

(i) *Given* that $\rho_t + c\rho_x = 0$, show that $c_t + c\, c_x = 0$. If $\rho(x, 0) = f(x)$, deduce that in regions where $c(x, t)$ is continuously differentiable:

$$c = c\{f(\xi)\} = F(\xi) \ \text{ on straight lines }\ x = \xi + F(\xi)t.$$

(13 marks)

 Question 5 continued on next page

5 (continued)

(ii) A shock occurs with values of $(\rho,\ \ q = q(\rho),\ \ c = c(\rho))$ on the two sides of the shock equal to $(\rho_1,\ q_1,\ c_1)$ and $(\rho_2,\ q_2,\ c_2)$.

Given that the speed U of the shock satisfies

$$U = \frac{q_1 - q_2}{\rho_1 - \rho_2},$$

show that

$$U = \frac{1}{2}(c_1 + c_2)$$

in the following two cases:

(a) exactly when $q(\rho)$ is a quadratic function of ρ;

(b) approximately when the shock is weak, i.e. when $|\rho_2 - \rho_1| \ll \rho_1$ and $|\rho_2 - \rho_1| \ll \rho_2$.

(12 marks)

End of Question Paper