The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2012-2013**

Milestones in Applied Mathematics II **2 Hours**

Marks will be awarded for your best FOUR answers.

1 For the following questions, you are given that the value of Planck's constant is $h = 6.626 \cdot 10^{-34}$Js and $\hbar = h/2\pi = 1.0546 \cdot 10^{-34}$Js. The speed of light is $c = 2.998 \cdot 10^{8}$m/s.

(i) A radio emitter radiates at a power of 10 kW at a wavelength of 100 m. How many photons does it emit per second? ***(4 marks)***

(ii) Consider a thermal neutron of mass $m_n = 1.67 \cdot 10^{-27} kg$ at temperature $T = 300K$. You are given that the energy of the neutron is $E = \frac{3}{2} k_b T$, where the Boltzmann constant is $k_b = 1.38 \cdot 10^{-23}$ J/K. Calculate the de-Brogie wavelength of the neutron. ***(6 marks)***

(iii) Show that if $\Psi_1(x,t)$ and $\Psi_2(x,t)$ are both solutions to Schrödinger's equation, then $\Psi_1(x,t) + \Psi_2(x,t)$ is a solution to Schrödinger's equation as well. ***(7 marks)***

(iv) The wavefunction of a free particle of mass m is given by

$$\Psi(x,t) = Ae^{i(kx-\omega t)},$$

where A is a complex constant, and k and ω are real constants. Find the energy and the momentum of the particle, and show that $\omega = \frac{\hbar k^2}{2m}$. ***(8 marks)***

2 (i) A particle of mass m moves freely in the interval $[0, a]$ on the x-axis. You are given that the wave-function solutions of the time-independent Schrödinger equation are given by

$$\Psi_n(x) = \begin{cases} A_n \sin\left(\dfrac{n\pi x}{a}\right), & 0 < x < a, \\ 0, & \text{otherwise,} \end{cases}$$

where the constants A_n are real and positive. Normalize Ψ_n, i.e.

$$\int_{-\infty}^{+\infty} \Psi_n^*(x)\Psi_n(x)dx = 1,$$

to find the constants A_n. ***(7 marks)***

(ii) Prove that for commutators A, B and C the following identities hold

(a) $[A + B, C] = [A, C] + [B, C]$ ***(3 marks)***

(b) $[A, BC] = [A, B]C + B[A, C]$ ***(3 marks)***

(iii) Let A and B be self-adjoint operators. Show that AB and BA are not self-adjoint in general, but that $AB + BA$ and $i(AB - BA)$ are both self-adjoint. ***(12 marks)***

3 (i) A particle of mass m is moving in one dimension and is described by a wavefunction $\Psi(x,t)$. The function $\Psi(x,t)$ is square-integrable, normalised and fulfills Schrödinger's equation. The expectation value for the position of the particle is

$$< x >= \int_{-\infty}^{+\infty} \Psi^*(x,t) x \Psi(x,t) dx.$$

The expectation value for the momentum of the particle is

$$< p >= \int_{-\infty}^{+\infty} \Psi^*(x,t) \frac{\hbar}{i} \frac{d}{dx} \Psi(x,t) dx.$$

Show that

$$\int_{-\infty}^{+\infty} j(x) dx = \frac{< p >}{m},$$

where $j(x)$ is the probability current. ***(8 marks)***

(ii) Show that

$$\frac{d}{dt} < x >= \frac{< p >}{m}.$$

[Hint: In both parts, you need to perform partial integration. The function Ψ is square-integrable, this means that $|\Psi(x,t)|^2$ vanishes for $x \to \pm\infty$. Likewise, $d\Psi/dx = 0$ and $d\Psi^*/dx = 0$ for $x \to \pm\infty$. Assume that

$$x\Psi \frac{d\Psi^*}{dx} \qquad \text{and}$$
$$x\Psi^* \frac{d\Psi}{dx}$$

vanish for $x \to \pm\infty$.] ***(17 marks)***

4 Consider the step potential

$$V(x) = \begin{cases} V_0, & x > 0 \quad \text{(Region I)}; \\ 0, & \text{otherwise} \quad \text{(Region II)}. \end{cases}$$

A current of particles of energy $E > V_0$ is moving from $x = -\infty$ to $x = +\infty$. You are *given* that the general stationary solution in region I is

$$\phi_I(x) = A_1 e^{ik_1 x} + A_1' e^{-ik_1 x}$$

with $k_1 = \sqrt{\dfrac{2m(E - V_0)}{\hbar^2}}$ and that in region II the general stationary solution is

$$\phi_{II}(x) = A_2 e^{ik_2 x} + A_2' e^{-ik_2 x}$$

with $k_2 = \sqrt{\dfrac{2mE}{\hbar^2}}$.

(i) Suppose that $A_1' = 0$. Use the matching condition to show that

$$\frac{A_1}{A_2} = \frac{2k_2}{k_1 + k_2}.$$

(8 marks)

(ii) Suppose that $A_1' = 0$. Show that the probability currents in regions I and II are given by

$$J_I(x) = \frac{\hbar k_1}{m} |A_1|^2$$

and

$$J_{II}(x) = \frac{\hbar k_2}{m} \left(|A_2|^2 - |A_2'|^2 \right)$$

respectively. ***(17 marks)***

5 An important operator in quantum computing is the Hadamard operator A (also called the Hadamard gate). It is, in a certain basis, given by

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

(i) Show that A is self-adjoint and that $\bar{A}^T A = \mathbf{1}$, where $\mathbf{1}$ is the unit matrix. ***(5 marks)***

(ii) Find the eigenvalues and normalized eigenvectors of A. ***(16 marks)***

(iii) The states Ψ_0 and Ψ_1 are defined as

$$\Psi_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \Psi_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Show that

$$\begin{aligned} A\left(\frac{1}{\sqrt{2}}\Psi_0 - \frac{1}{\sqrt{2}}\Psi_1\right) &= \Psi_1 \qquad \text{and} \\ A\left(\frac{1}{\sqrt{2}}\Psi_0 + \frac{1}{\sqrt{2}}\Psi_1\right) &= \Psi_0. \end{aligned}$$

(4 marks)

End of Question Paper