The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2012–13**

Topics in Number Theory (Level 3) **2 hours 30 minutes**

Answer ***four*** *questions. If you answer more than four questions, only your best four will be counted.*

No credit will be given for solutions which rely solely on the use of a calculator. Your solutions should give enough details to make it clear how you arrived at your answers.

1 (i) You publish $(n, e) = (851, 97)$ in the RSA directory and receive 3. Decode it. ***(10 marks)***

(ii) (a) Prove that $a^{1105} \equiv a \pmod{1105}$ for all integers a. ***(6 marks)***

(b) The integer $n > 1$ is such that $a^{n-1} \equiv 1 \pmod{n}$ for all positive integers a smaller than n. Prove that n must be prime. ***(4 marks)***

(iii) Find the remainder when 39! is divided by 215. (No credit will be given for a solution which does not use Wilson's theorem.) ***(5 marks)***

2 (i) State *Euler's criterion.* ***(2 marks)***

(ii) State the *Law of Quadratic Reciprocity.* ***(2 marks)***

(iii) Let $p > 3$ be a prime number. Show that the congruence

$$x^4 + 7x^2 + 12 \equiv 0 \pmod{p}$$

does *not* have a solution if and only if $p \equiv 11 \pmod{12}$, and find all the solutions in the case $p = 43$. ***(13 marks)***

(iv) Use the fact that for any prime number $p > 3$

$$\left(\frac{3}{p}\right) = \begin{cases} 1 & \text{if } p \equiv 1 \text{ or } 11 \pmod{12} \\ -1 & \text{if } p \equiv 5 \text{ or } 7 \pmod{12} \end{cases}$$

to prove that there exist infinitely many primes of the form $12k + 11$. ***(8 marks)***

3 (i) Show that $2^{13} - 1$ is prime. ***(7 marks)***

(ii) Use the Law of Quadratic Reciprocity to prove that 3 is a quadratic non-residue modulo any Mersenne prime greater than 3. ***(5 marks)***

(iii) (a) Define a perfect number. ***(1 mark)***

(b) State a formula which gives all even perfect numbers, and prove that every number given by your formula is perfect. You should define any notation you use. ***(6 marks)***

(c) Prove that the *product* of the positive divisors of an even perfect number n is n to the power of a prime number. ***(6 marks)***

4 (i) State formulae which describe all Pythagorean triples (x, y, z), where the highest common factor of x, y, z is k. ***(3 marks)***

(ii) Determine the parameters in your formulae in your answer to (i) that produce the Pythagorean triple $(363, 1980, 2013)$. ***(3 marks)***

(iii) Determine all Pythagorean triples, not necessarily primitive, which include the number 671. ***(13 marks)***

(iv) Prove that if (x, y, z) is a Pythagorean triple with $x^2+y^2 = z^2$ then $x+y+z$ divides xy. ***(6 marks)***

5 (i) Express $\dfrac{35}{103}$ as a continued fraction. ***(4 marks)***

(ii) Express $[3; 2, 1, \overline{1, 2}]$ in the form $a + b\sqrt{c}$, where a, b are rational numbers and c is a positive integer. ***(7 marks)***

(iii) (a) Show that if n is a positive integer then $\sqrt{n^2 + 2n} = [n; \overline{1, 2n}]$. ***(7 marks)***

(b) Find a convergent of $\sqrt{15}$ which differs from it by less than 10^{-5} and find a solution to Pell's equation

$$x^2 - 15y^2 = 1$$

in which $x > 100$. ***(7 marks)***

End of Question Paper