The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2012–13**

Metric Spaces **2 hours 30 minutes**

*Answer **four** questions. If you answer more than four questions, only your best four will be counted.*

1 (i) Let $n \geqslant 1$. The metric d_∞ on $\mathbb{R}^n$ is such that, for all $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, $d_\infty(x, y) = \max(|x_1 - y_1|, \ldots, |x_n - y_n|)$. Show that d_∞ is indeed a metric on $\mathbb{R}^n$. ***(6 marks)***

(ii) Let (X, d) be a metric space, let $x \in X$ and let $r \geqslant 0$. Define the *closed ball $B[x, r]$ centred at x with radius r.* ***(2 marks)***

(iii) Consider the metric space $\mathbb{R}^2$ with the taxicab metric d_1, that is,

$$d_1((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|.$$

(a) Compute $d_1((0, 0), (1, 1))$. ***(1 mark)***

(b) Show that if $x \geqslant 0$, $y \geqslant 0$ and $(x, y) \in B[(0, 0), 1]$ then $x + y \leqslant 1$. ***(1 mark)***

(c) Show that if $(x, y) \in B[(1, 1), 1]$ then $x \geqslant 0$, $y \geqslant 0$ and $x + y \geqslant 1$. Deduce that if $(x, y) \in B[(0, 0), 1] \cap B[(1, 1), 1]$ then $y = 1 - x$. ***(6 marks)***

(d) Show that if $0 \leqslant x \leqslant 1$ then $(x, 1 - x) \in B[(0, 0), 1] \cap B[(1, 1), 1]$. ***(3 marks)***

(iv) For $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $(y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, let

$$e(x, y) = \max(|x_i - y_i|) + \min(|x_i - y_i|).$$

(a) Show that if $n = 2$ then, for all $x, y \in \mathbb{R}^n$, $e(x, y) = d_1(x, y)$. ***(3 marks)***

(b) By considering $(1, 1, 1), (1, 2, 3)$ and $(2, 2, 4)$, show that if $n = 3$ then e is not a metric. ***(3 marks)***

2 The metrics d_1 and d_∞ on the space $C[0,1]$ of continuous functions from $[0,1]$ to $\mathbb{R}$ are given by the rules

$$d_1(f,g) = \int_0^1 |f(x)-g(x)|\,dx, \quad d_\infty(f,g) = \sup_{0\le x\le 1} |f(x)-g(x)| \text{ for all } f,g \in C[0,1].$$

(i) Let $a > 0$ and let $f \in C[0,1]$ be the function such that $f(x) = 0$ for all $x \in [0,1]$. Compute $d_1(e^{-ax}, f)$ and $d_1(e^{-ax}, e^{ax})$. ***(4 marks)***

Show that e^{-ax} is strictly decreasing on $[0,1]$ and hence find $d_\infty(e^{-ax}, f)$. Also find $d_\infty(e^{-ax}, e^{ax})$. ***(4 marks)***

(ii) For $n \geqslant 1$, let $f_n \in C[0,1]$ be the function such that $f_n(x) = e^{-nx}$ for all $x \in [0,1]$.

(a) Show that $f_n \to f$ in $(C[0,1], d_1)$. ***(4 marks)***

(b) Explain why $f_n(0) \to 1$ in $\mathbb{R}$ with its usual metric. Deduce that the function $\theta : C[0,1] \to \mathbb{R}$ given by $\theta(g) = g(0)$ is not continuous when $C[0,1]$ has the metric d_1 and $\mathbb{R}$ has its usual metric. ***(4 marks)***

(iii) (a) Show that if a sequence (f_n) of functions converges to f in $(C(I), d_\infty)$, for some closed interval I, then (f_n) converges to f pointwise on I. ***(4 marks)***

(b) Does the sequence (f_n) specified in (ii) converge pointwise to a function in $C[0,1]$? Does the sequence (f_n) converge in $(C[0,1], d_\infty)$? Justify your answers. ***(5 marks)***

3 (i) Let (X,d) be a metric space and let A be a subset of X.

(a) Explain what it means for A to be *open* and what it means for A to be *closed*. ***(4 marks)***

(b) Show that A is open if and only if the complement $X\backslash A$ is closed. ***(8 marks)***

(ii) Consider $\mathbb{R}^3$ with the Euclidean metric d_2. Let

$$\begin{aligned} A_1 &= \{(x,y,z) : x,y,z \in \mathbb{R} \text{ and } z \neq 0\}, \\ A_2 &= \{(x,y,0) : x,y \in \mathbb{R}\}. \end{aligned}$$

(a) Show that $\left(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}\right) \to (0,0,0)$ in $(\mathbb{R}^3, d_2)$. ***(3 marks)***

(b) Let $((x_n, y_n, 0))$ be a convergent sequence in $\mathbb{R}^3$ with limit (x,y,z). Show that $d_2((x_n,y_n,0),(x,y,z)) \geqslant |z|$ for all n and deduce that $z = 0$. ***(3 marks)***

(c) Is A_1 closed? Is A_2 closed? Is A_1 open? Is A_2 open? Justify your answers. ***(7 marks)***

4 (i) Define a *contraction* on a metric space (X, d) and state, without proof, the Contraction Mapping Principle. ***(4 marks)***

(ii) Let I be a closed interval which may be bounded or unbounded. For a function $f : I \to I$ let $(*)$ denote the property that

$$|f(x) - f(y)| < |x - y| \text{ for all } x \neq y \in I.$$

(a) Show that the function $g(x) = x + \dfrac{1}{x}$ has the property $(*)$ on $[1, \infty)$. ***(4 marks)***

Explain why g cannot be a contraction on $[1, \infty)$. ***(3 marks)***

(b) Show that the function $h(x) = \cos(x)$ has the property $(*)$ on $\mathbb{R}$. You may assume that $|\sin\theta| < |\theta|$ whenever $\theta \neq 0$. ***(4 marks)***

(c) For $x \in [1, \infty)$, let

$$j(x) = \frac{1}{2}\left(x + \frac{1}{x}\right) + \frac{3 + \cos x}{4}.$$

Show that $j(x) \in [1, \infty)$ for all $x \in [1, \infty)$ and that the function $j : [1, \infty) \to [1, \infty)$ is a contraction. ***(7 marks)***

Deduce that there is a unique number $x \in [1, \infty)$ such that

$$x \cos x = (x - 2)(2x + 1).$$

(3 marks)

5 (i) Let (X,d) be a metric space.

(a) Define a *Cauchy sequence* (x_n) in (X,d). ***(2 marks)***

(b) Define a *complete* metric space (X,d). ***(1 mark)***

(c) Show that every convergent sequence in (X,d) is a Cauchy sequence. ***(4 marks)***

(ii) Let (f_n) be the sequence in $(C[0,\frac{1}{2}],d_\infty)$ such that, for each $x \in [0,\frac{1}{2}]$,

$$f_n(x) = 1 + x + x^2 + \cdots + x^n.$$

(a) For $m > n$, show that $f_m(x) - f_n(x) \geqslant 0$ for all $x \in [0,\frac{1}{2}]$ and that $f_m - f_n$ is strictly increasing on $(0,\frac{1}{2}]$. Hence show that, for $m > n$,

$$d_\infty(f_n,f_m) = f_m\left(\frac{1}{2}\right) - f_n\left(\frac{1}{2}\right) = \frac{1}{2^n} - \frac{1}{2^m}.$$

(6 marks)

(b) Show that (f_n) is a Cauchy sequence. ***(3 marks)***

(c) Does (f_n) converge in $(C[0,\frac{1}{2}],d_\infty)$? Justify your answer. ***(2 marks)***

(iii) Let (x_n) be a Cauchy sequence in a metric space (X,d) and let $a \in X$. Show that if (x_n) has a subsequence (x_{n_k}) that converges to a then (x_n) converges to a. Deduce that if (X,d) is a compact metric space then it is complete. ***(7 marks)***

End of Question Paper