The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2012-2013**

Complex Analysis **2 hours 30 minutes**

*Answer **four** questions. If you answer more than four questions, only your best four will be counted.*

1 (i) Express both of the following in the form $x+iy$:

$$\frac{17+9i}{1+2i}\,; \qquad \left(\sqrt{3}+i\right)^{13}.$$ ***(4 marks)***

(ii) Express

$$\frac{(\sqrt{3}+i)^7}{(1-i)^{17}}$$

in the form $re^{i\theta}$ with $r>0$ and $-\pi<\theta\le\pi$. ***(4 marks)***

(iii) State, without proof, the triangle inequalities for $|z+w|$ and $|z-w|$.

Show that, if $|z-1-i| \le 2$, then

$$4 \le |3z+3+5i| \le 16\,.$$ ***(4 marks)***

(iv) Find all the solutions of each of the following equations:

(a) $e^{2iz}+1=0$,

(b) $\cosh z = i$. ***(7 marks)***

(v) The path γ is the arc of the circle $|z-1| = 2$ from 3 to -1 given by $z=1+2e^{it}$ $(0\le t\le\pi)$. Evaluate

$$\int_\gamma \mathrm{Re}\,z\,dz\,, \qquad \int_\gamma z\,e^{z^2}\,dz\,.$$ ***(6 marks)***

2 (i) Define what is meant by the following two statements:
(a) A function f is **differentiable at the point** z_0 ;
(b) A function f is **analytic in a region** D. ***(2 marks)***

Let

$$g(z) = \frac{\cosh(z)}{(1+e^z)^3} \ .$$

Decide where g is analytic giving reasons for your answer. ***(4 marks)***

(ii) State, without proof, the Cauchy-Riemann equations for a differentiable function. ***(1 mark)***

Let

$$h(z) \ = \ |z|^2.$$

Prove that, if $z_0 \ \neq \ 0$, then the function h is not differentiable at z_0. Is h differentiable at the origin? Give reasons for your answer. ***(4 marks)***

(iii) Find all the functions k analytic on $\mathbb{C}$ with $\text{Re}\,(k(x+iy)) = 2x - \sinh x \sin y$, giving an explicit expression for $k(z)$ in terms of z. Show that you have found **all** the functions satisfying the above conditions. ***(6 marks)***

(iv) Let D be the disc $D = \{z \in \mathbb{C} : |z+1| < 3\}$. Sketch D and show that

$$\left| \int_\gamma \frac{ze^z}{z^2+16}\, dz \right| \ \leq \ \frac{3\sqrt{10}}{7}$$

for **all paths** γ in D with initial point -3 and final point i. ***(8 marks)***

3 State, without proof, Cauchy's Theorem and Cauchy's Integral Formulae for a function and for its derivatives. Your statement should include conditions under which the results are valid. ***(7 marks)***

Let γ be the circular contour $|z-1| = 3$ described in the anti-clockwise direction. Without using the Residue Theorem, evaluate

(i) $\displaystyle\int_\gamma \frac{\cos z}{z}\, dz$, (ii) $\displaystyle\int_\gamma \frac{\sin(\pi z)}{z^2-9}\, dz$,

(iii) $\displaystyle\int_\gamma \frac{\cosh z}{z^2+16}\, dz$, (iv) $\displaystyle\int_\gamma \frac{e^{3z}}{(2z+1)^3}\, dz$,

(v) $\displaystyle\int_\gamma \operatorname{Im} z\, dz$.

(18 marks)

4 (i) Find the radius of convergence of the following power series :

$$\sum_{n=1}^{\infty} \left(4^n - (-3)^n\right) z^n.$$ ***(2 marks)***

(ii) For each of the following functions, find **all the singularities** in $\mathbb{C}$. Classify these singularities giving reasons for your answers and evaluate the residue at each of them:

(a) $\dfrac{\sin(\pi z)}{z-1}$; ***(3 marks)***

(b) $z \sin\left(\dfrac{1}{z+1}\right)$; ***(4 marks)***

(c) $\dfrac{z}{e^z - 1}$; ***(7 marks)***

(d) $\dfrac{e^{\pi z}}{(z-1)^9}$; ***(5 marks)***

(e) $\dfrac{1-\cos z}{z^5}$. ***(4 marks)***

5 (i) State, without proof, Cauchy's Residue Theorem. Your statement should include conditions under which the result is valid. ***(4 marks)***

Let γ be the circular contour $|z| = 1$ described in the anti-clockwise direction. Evaluate

$$\int_\gamma \frac{z}{\cos \pi z} \, dz \, , \qquad \int_\gamma \frac{\sin \pi z}{(2z-1)(z^2+9)} \, dz \, .$$

(12 marks)

(ii) Prove that

$$\int_{-\infty}^{\infty} \frac{\cos \pi x}{z^2 - 2z + 5} \, dx \quad = \quad - \, \frac{\pi}{2} \, e^{-2\pi} \, .$$

(9 marks)

End of Question Paper