The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Autumn Semester 2012-13

Fields - MAS333 — **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

Please leave this exam paper on your desk
Do not remove it from the hall

Registration number from U-Card (9 digits)
to be completed by student

Blank

1 (i) For each of the subsets J_1, J_2 of $\mathbb{C}$ specified below determine, with justification, whether it is a subfield of $\mathbb{C}$:

(a) $J_1 = \{a + b\sqrt{3} : a, b \in \mathbb{Q}\}$, ***(5 marks)***

(b) $J_2 = \{a + b\sqrt{3} + ci : a, b, c \in \mathbb{Q}\}$. ***(3 marks)***

(ii) Let K and L be fields such that $[K : \mathbb{Q}] = 5$ and $[L : \mathbb{Q}] = 4$. Is the following situation possible $K \subseteq L$? Justify your answer. ***(4 marks)***

(iii) Find the subfield of $\mathbb{C}$ generated by the numbers $\{\sqrt{2}, \sqrt{3}\}$ and give a possible $\mathbb{Q}$-basis. Justify your answer. ***(9 marks)***

(iv) Let K be a subfield of a field L. Give a definition of $[L : K]$. ***(2 marks)***

(v) Express the complex number $\dfrac{(1-2i)(1+2i)}{1-i}$ in the form $a + bi$ where $a, b \in \mathbb{R}$. ***(2 marks)***

2 (i) State Gauss's Lemma. ***(3 marks)***

(ii) Prove Gauss's Lemma. ***(8 marks)***

(iii) Give a definition of the content $c(f)$ of a polynomial $f \in \mathbb{Z}[x]$. ***(2 marks)***

(iv) Explain what the multiplicativity of the content means. ***(2 marks)***

(v) Write $x^n - 1$ as a product of the cyclotomic polynomials $\phi_m(x)$ (no proof is required). ***(3 marks)***

(vi) Give a proof of the expression you wrote down in 2(v). ***(7 marks)***

3 (i) Let $K \subseteq L$ be a field extension, and $a \in L$.

(a) Explain what it means for the element a to be algebraic over the field K. ***(2 marks)***

(b) Give the definition of the minimal polynomial $m(x) \in K[x]$ of the algebraic element a over the field K. ***(3 marks)***

(c) Prove that the minimal polynomial $m(x)$ is an irreducible polynomial over the field K. ***(6 marks)***

(d) Prove that if the element a is a root of a polynomial $p(x) \in K[x]$ then $p(x) = m(x)q(x)$ for some polynomial $q(x) \in K[x]$. ***(6 marks)***

(ii) Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in \mathbb{Z}[x]$ be the polynomial with integer coefficients such that $a_n \neq 0$, $a_0 \neq 0$ and there exists a prime number p such that $p|a_i$ for all $i = 1, \ldots n$; $p \nmid a_0$ and $p^2 \nmid a_n$. Prove that the polynomial $f(x)$ is an irreducible polynomial in $\mathbb{Q}[x]$. ***(8 marks)***

4 (i) Let B be a set of (at least 2) points in the plane $\mathbb{R}^2$, and $P, Q \in \mathbb{R}^2$. Explain what does it mean that the point P is **constructible in one step from** B and the point Q is **constructible from** B. ***(4 marks)***

(ii) Define the set of constructible points. ***(2 marks)***

.

(iii) Let $(a, b) \in \mathbb{R}^2$. Give a criterion for constructibility of the point (a, b) (via quadratic fields). ***(3 marks)***

(iv) Prove the criterion. ***(8 marks)***

(v) Using **only** the criterion show that the point $(\sqrt{2+\sqrt[4]{3}}, 0)$ is a constructible point. ***(4 marks)***

(vi) Is the point $(\frac{1}{2}, \sqrt[3]{2})$ constructible? Justify your response. ***(4 marks)***

End of Question Paper