The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2012–13**

Combinatorics **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) (a) State the Binomial Theorem. ***(2 marks)***

(b) By integrating, or otherwise, show that

$$\sum_{i=0}^{n} \frac{\binom{n}{i}}{i+1} = \frac{2^{n+1}-1}{n+1}.$$

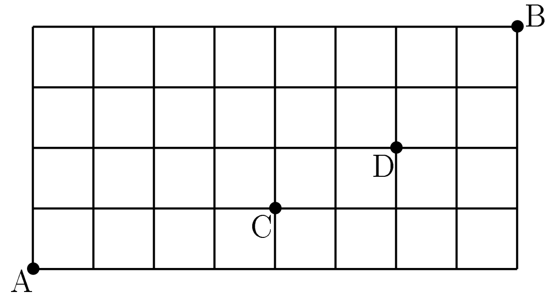
(6 marks)

(ii) Suppose there are $2n$ pupils, n from School A and n from School B. We want to form a team of n players from these $2n$ pupils, with one of the pupils from School A as leader of the team. By considering the number of ways to do this, show that

$$\sum_{k=1}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}.$$

(8 marks)

(iii) This part of the question concerns routes in the grid illustrated:



(a) How many shortest routes are there from A to B along the lines of the grid? Give a brief reason for your answer. ***(3 marks)***

(b) Find the number of such routes which do not pass through C or D. ***(6 marks)***

2 (i) (a) State the Pigeon-hole Principle. ***(2 marks)***

(b) Show that if X is a set of nine different integers from 1 to 60, there are two disjoint subsets of X with the same sum. ***(6 marks)***

(ii) (a) Consider the equation

$$x_1 + x_2 + \cdots + x_k = n.$$

How many solutions are there of this equation in which each x_i is a non-negative integer? Give a brief reason for your answer. ***(3 marks)***

(b) How many non-negative integer solutions are there of

$$x_1 + x_2 + x_3 + x_4 \leqslant 10 \quad ?$$

(3 marks)

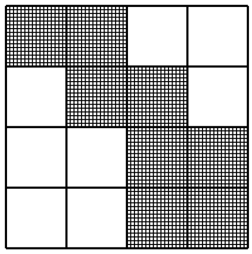
(c) State the Inclusion/Exclusion Principle. ***(3 marks)***

(d) Use the Inclusion/Exclusion Principle to find the number of non-negative integer solutions of the equation

$$x_1 + x_2 + x_3 + x_4 = 10,$$

satisfying the conditions $x_1 < 3$, $x_2 < 6$ and $x_3 < 7$. ***(8 marks)***

3 (i) Calculate the rook polynomial of the board:



(6 marks)

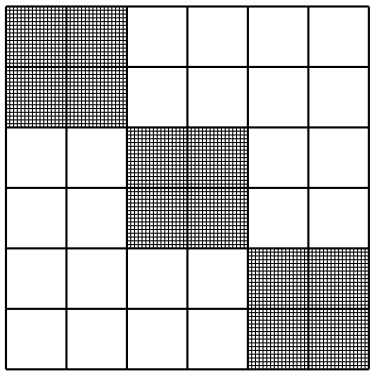
(ii) Which of the following polynomials can be the rook polynomial of a board? Give reasons for your answers, including examples of appropriate boards where relevant.

(a) $1 - 2x$.

(b) $(1 + x)^2(1 + 4x + 2x^2)$.

(c) $1 + 4x + 7x^2 + 2x^3 + x^4$. ***(5 marks)***

(iii) In how many ways can 6 non-challenging rooks be placed on the following board?



(7 marks)

(iv) (a) Explain what is meant by a tournament of n players. ***(2 marks)***

(b) If $w_1, w_2, \ldots, w_n$ are the scores of a tournament of n players, show that $n - 1 - w_n, n - 1 - w_{n-1}, \ldots, n - 1 - w_1$ are also scores of a tournament of n players. ***(5 marks)***

4 (i) State necessary and sufficient conditions for a $p \times q$ Latin rectangle to be extendable to an $n \times n$ Latin square. ***(2 marks)***

(ii) For what value of x can the following Latin rectangle be extended to a 6×6 Latin square?

$$\begin{pmatrix} 1 & 3 & 2 & x \\ 5 & 1 & 6 & 2 \\ 6 & 4 & 3 & 1 \\ 2 & 5 & 1 & 3 \end{pmatrix}$$

Write down one such extension. ***(8 marks)***

(iii) (a) In a (v, b, r, k, λ) design, v is the number of varieties and b is the number of blocks. Explain the meanings of r, k and λ. ***(3 marks)***

(b) Prove that, in a (v, b, r, k, λ) design,

$$r = \frac{bk}{v} = \frac{\lambda(v-1)}{k-1}.$$

(7 marks)

(c) Explain what it means for a (v, b, r, k, λ) design to be symmetric. ***(2 marks)***

(d) Consider a symmetric (v, b, r, k, λ) design. Show that $k + \lambda(v-1)$ is a square number. ***(3 marks)***

End of Question Paper