The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2012–13**

Differential Geometry **2 hours 30 minutes**

Answer ***four*** *questions. If you answer more than four questions, only your best four will be counted.*

A list of formulae is provided on the last page.

1 (i) Let **Heli** be the surface parametrised by

$$\mathbf{x}(u,v) = (v\cos u, v\sin u, u), \qquad u, v \in \mathbb{R}.$$

The first and second fundamental forms of $\mathbf{x}(u,v)$ are respectively

$$(1+v^2)du^2 + dv^2 \qquad \text{and} \qquad 2(1+v^2)^{-1/2}dudv.$$

(a) Find the Weingarten matrix of $\mathbf{x}(u,v)$. ***(5 marks)***

(b) Show that the principal curvatures are $\pm(1+v^2)^{-1}$. ***(5 marks)***

(c) For each principal curvature, find a corresponding principal vector. ***(8 marks)***

(ii) Prove that, if a surface in $\mathbb{R}^3$ contains a straight line, then its Gaussian curvature at every point on the line is either zero or negative. ***(7 marks)***

2 (i) Let **Heli** be (again) the surface parametrised by

$$\mathbf{x}(u, v) = (v \cos u, v \sin u, u), \qquad u, v \in \mathbb{R}.$$

Define a curve on **Heli** by $\gamma(t) = \mathbf{x}(t, 0)$ for $t \in \mathbb{R}$. Also define a vector-valued function by $\mathbf{v}(t) = (\cos t, \sin t, 1)$ for $t \in \mathbb{R}$.

(a) Find a normal vector of **Heli** at $\gamma(t)$ for each t. ***(4 marks)***

(b) Show that $\mathbf{v}(t)$ is a vector field of **Heli** along $\gamma(t)$. ***(4 marks)***

(c) Show that the vector field in (b) is parallel. ***(4 marks)***

(d) Suppose $\mathbf{w}(t)$ is another parallel vector field on S along $\gamma(t)$. Prove that the angle between $\mathbf{v}(t)$ and $\mathbf{w}(t)$ is independent of t. ***(5 marks)***

(ii) Let $\gamma(s)$ be a unit-speed curve on a surface $S \subset \mathbb{R}^3$. Suppose there is a constant nonzero vector $\mathbf{a}$ such that $\gamma'(s)$ is always orthogonal to $\mathbf{a}$, and the normal vectors of S along $\gamma(s)$ are also orthogonal to $\mathbf{a}$. Prove that $\gamma(s)$ is a geodesic on S. ***(8 marks)***

3 (i) The parabola $z = x^2$ on the xz-plane is rotated around the z-axis to produce a surface of revolution. Calculate the area of the region below $z = 1$. ***(10 marks)***

(ii) Let $A = \{(x, y) \in \mathbb{R}^2 \,|\, x > 0\}$. The parametrisation of A using polar coordinates, namely

$$(r \cos \theta, r \sin \theta), \qquad r > 0, \;\; \theta \in (-\pi/2, \pi/2),$$

has the first fundamental form $dr^2 + r^2 d\theta^2$. Let **Cone** be the surface of revolution obtained by rotating the straight line $z = \sqrt{3}x$ on the xz-plane around the z-axis. Define a map $\varphi : A \to$ **Cone** by

$$\varphi(r \cos \theta, r \sin \theta) = (ar \cos b\theta, ar \sin b\theta, \sqrt{3}ar),$$

where a and b are some positive constants.

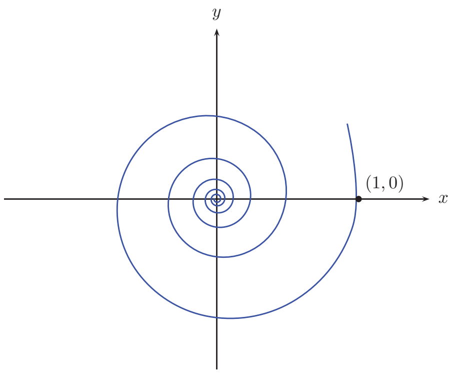
(a) Find the values of a and b such that φ is a local isometry. ***(8 marks)***

(b) In the case φ is a local isometry, find the length of the shortest path on **Cone** between $(1, 0, \sqrt{3})$ and $(0, 1, \sqrt{3})$. ***(7 marks)***

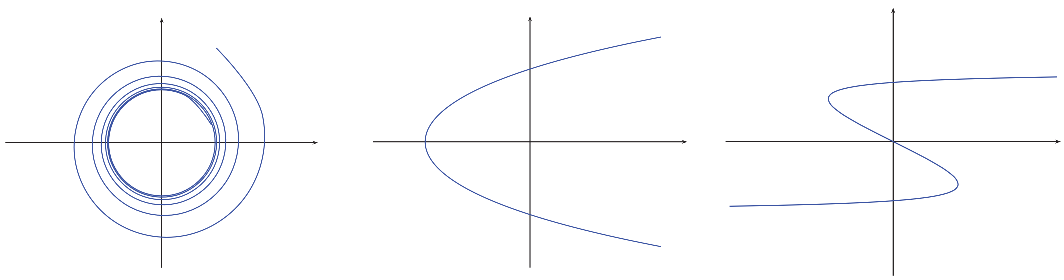
4 (i) Let a be a positive number. Consider the curve parametrised by

$$(e^{at}\cos t, e^{at}\sin t), \qquad t \in \mathbb{R}.$$

Find its arc length between $(1,0)$ and the origin (see picture). ***(7 marks)***



(ii) Among the curves in the following pictures, one of them has a unit-speed parametrisation $\mathbf{x}(s)$ whose curvature function is $k(s) = (1+s^2)^{-1}$. Which one is it? Explain your answer. ***(5 marks)***



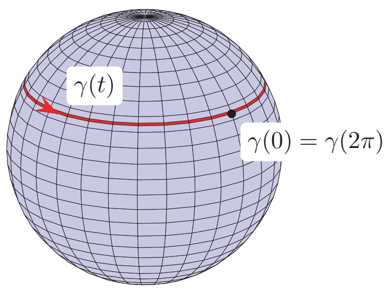
(iii) Suppose the curvature of a curve parametrised by $\mathbf{x}(t)$ is $k(t)$. Let c be a positive number. What is the curvature of the curve parametrised by $c\mathbf{x}(t)$? Explain your answer in details. ***(6 marks)***

(iv) Prove that a curve on $\mathbb{R}^2$ with *constant* positive curvature must be contained in a circle. ***(7 marks)***

5 (i) Consider the sphere $S^2 \subset \mathbb{R}^3$ with the parametrisation

$$\mathbf{x}(\phi, \theta) = (\cos\phi\cos\theta, \cos\phi\sin\theta, \sin\phi).$$

The first fundamental form of $\mathbf{x}(\phi, \theta)$ is $d\phi^2 + \cos^2\phi\, d\theta^2$. The Gaussian curvature has value 1 at every point. Also consider the curve (latitude) on S^2 parametrised by $\gamma(t) = \mathbf{x}(\alpha, t)$, where α is some constant.



(a) Let $\mathbf{v}(t)$ for $t \in \mathbb{R}$ be a parallel vector field along $\gamma(t)$. Show that the angle between $\mathbf{v}(0)$ and $\mathbf{v}(2\pi)$ is $2\pi(1 - \sin\alpha)$. ***(6 marks)***

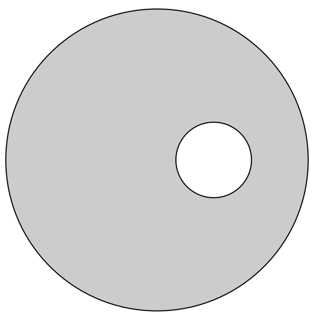
(b) By symmetry, the curve parametrised by $\gamma(t)$ has constant geodesic curvature. Find its value. ***(7 marks)***

(c) Show that for any geodesic triangle Δ, there is an identity

$$\text{sum of internal angles in } \Delta = \pi + \text{area of } \Delta.$$

(7 marks)

(ii) Using a triangulation, find the Euler characteristic of a disk with a hole. ***(5 marks)***



End of Question Paper

List of Formulae

For a curve on $\mathbb{R}^2$ parametrised by $\mathbf{x}(t) = (x(t), y(t))$:

- arc length from $\mathbf{x}(a)$ to $\mathbf{x}(b)$

$$\int_a^b ||\mathbf{x}'(t)||\,dt$$

- curvature

$$k(t) = \frac{x'(t)y''(t) - y'(t)x''(t)}{[x'(t)^2 + y'(t)^2]^{3/2}}$$

For a surface in $\mathbb{R}^3$ parametrised by $\mathbf{x}(u, v)$:

- first fundamental form

$$Edu^2 + 2Fdudv + Gdv^2, \quad E = \mathbf{x}_u \cdot \mathbf{x}_u,\ F = \mathbf{x}_u \cdot \mathbf{x}_v,\ G = \mathbf{x}_v \cdot \mathbf{x}_v$$

- surface areas

$$\iint \sqrt{EG - F^2}\,dudv$$

- second fundamental form

$$Ldu^2 + 2Mdudv + Ndv^2, \quad L = \mathbf{x}_{uu} \cdot \mathbf{n},\ M = \mathbf{x}_{uv} \cdot \mathbf{n},\ N = \mathbf{x}_{vv} \cdot \mathbf{n}$$

$$\text{where } \mathbf{n} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{||\mathbf{x}_u \times \mathbf{x}_v||}$$

- Weingarten matrix

$$W = \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \begin{bmatrix} L & M \\ M & N \end{bmatrix}$$

- mean and Gaussian curvatures

$$H = \frac{1}{2}\operatorname{tr} W, \qquad K = \det W$$

The Gauss-Bonnet formula for a compact region R on a surface:

$$\iint_R K dA + \int_{\partial R} k_g ds + \sum \text{turning angles} = 2\pi\chi(R)$$