The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2012-2013**

Mathematics (Computational Methods) **2 hours**

Answer ***four*** *questions. If you answer more than four questions, only your best four will be counted.*
Graph paper is provided for question 5.

1.

Let $$A=\begin{pmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{pmatrix}.$$

(i) Find the LU decomposition of A, where L is a lower triangular matrix with ones on the principal diagonal and U is an upper triangular matrix. **(*6 marks*)**

(ii) Verify that L^{-1} and U^{-1} have, respectively, the forms:

$$L^{-1}=\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix}, \quad U^{-1}=\begin{pmatrix} \frac{1}{4} & \frac{1}{15} & d \\ 0 & \frac{4}{15} & \frac{1}{14} \\ 0 & 0 & \frac{15}{56} \end{pmatrix}$$

and find the values of a, b, c and d. **(*6 marks*)**

(iii) Explain how you would use the result of part (ii) to find A^{-1}. Given that it has the form:

$$A^{-1}=\frac{1}{56}\begin{pmatrix} 15 & 4 & 1 \\ 4 & e & 4 \\ 1 & 4 & 15 \end{pmatrix},$$

find the value of e. **(*3 marks*)**

(iv) The Crank-Nicolson scheme for finding the approximate solution of the heat equation

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2},$$

where $u(0,t)=14, u(1,t)=0$ and $u(x,0)=21-28\left|x-\frac{1}{4}\right|$, is

$$-ru_{i-1,j+1}+2(1+r)u_{i,j+1}-ru_{i+1,j+1}=ru_{i-1,j}+2(1-r)u_{i,j}+ru_{i+1,j}+O\left(k^3,kh^2\right),$$

where $r=k/h^2$ and $u_{i,j}=u(ih,jk)$. Letting $h=0.25, k=0.0625$, set up a table showing the values of u at the grid points for $t=0$, and hence calculate the values of u at the grid points for $t=0.0625$. **(*10 marks*)**

2.

(i) Let $A = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 7 & 3 & 1 \\ 3 & 9 & 3 \\ 2 & 6 & 8 \end{pmatrix}$.

Evaluate AB and hence or otherwise find A^{-1}.

(*4 marks*)

(ii) The solution of the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is to be approximated in the square region:

$$\left\{(x, y) : 0 \le x \le \frac{3}{2},\ 0 \le y \le \frac{3}{2}\right\},$$

subject to the boundary conditions:

$$u = 3 + 3y - 2y^2 \text{ when } x = 0,$$
$$u = 3 + x - 2x^2 \text{ when } y = 0 \text{ or } y = \frac{3}{2},$$
$$\frac{\partial u}{\partial x} = -5 \text{ when } x = \frac{3}{2}.$$

Taking $h = k = \frac{1}{2}$, draw up a suitable grid for a numerical analysis of the problem, and mark on it known values of u. Also indicate on your diagram an appropriate notation for the unknown values and any fictitious values you will require to use.

(*6 marks*)

(iii) Write down equations relating the variables specified in part (ii) and, by eliminating any fictitious values and making use of any symmetry in the diagram, show that they are of the form $A\mathbf{x} = \mathbf{b}$ where A is the matrix in part (i) and $\mathbf{u}$ and $\mathbf{b}$ are column vectors. Hence solve the equations for the unknown values of $\mathbf{u}$.

(*15 marks*)

3.

(i) Give a brief description of a cubic spline, including a clear statement of the conditions that must be satisfied by the two formulae on either side of a datum point. (***6 marks***)

(ii) By deriving appropriate formulae from the requirements detailed in part (i), find the cubic spline which fits the following set of data:

x	0	1	2	3
$f(x)$	15	35	25	40

subject to the additional requirement that the tangent at each end should be horizontal. (***19 marks***)

You may use without proof that the inverse of the matrix

$$\begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \quad \text{is} \quad \frac{1}{45}\begin{pmatrix} 26 & -7 & 2 & -1 \\ -7 & 14 & -4 & 2 \\ 2 & -4 & 14 & -7 \\ -1 & 2 & -7 & 26 \end{pmatrix}.$$

4. Let $f(x,y) = 3x^2 + 3xy + y^2 - 6x - y + 8$.

(i) By finding the critical point directly, find the minimum value of $f(x,y)$. Use a second derivative test to verify that this is indeed a local minimum. (***8 marks***)

(ii) Starting from the initial point $(2,-1)$ use one iteration of the method of steepest descent to find a point where $f(x,y)$ has a lower value than at $(2,-1)$. Check that the gradient vector at this point is perpendicular to the gradient vector at $(2,-1)$. (***9 marks***)

(iii) Use one iteration of Newton's method, starting at $(2,-1)$ Check that the point obtained is the minimum point you found in part (i). (***8 marks***)

5. Illustrate the branch and bound method for solving integer programming problems on the following example:

Maximise $z = 7x + 8y$ subject to the constraints:

$x \geq 0, \quad y \geq 0, \quad 3x + 5y \leq 30, \quad 5x + 4y \leq 30, \quad x, y$ are integers,

It is not necessary to completely solve the problem, but you should find at least 7 nodes of the process. You may solve the associated linear programming problems by inspection of a graph, but the precise positions of the vertices should be calculated by solving appropriate simultaneous equations. You should also indicate the structure of your solution by drawing an appropriate tree diagram. (Simply evaluating the objective function at all feasible lattice points will gain very few marks)

(*25 marks*)

6. A container of volume $30\ \text{m}^3$ is to be loaded with a selection of goods of 3 types. These have volumes and values as shown in the following table:

Type	volume (m^3)	value (£'00)
1	9	22
2	7	18
3	5	13

Using the dynamic programming algorithm construct an appropriate table to find the combination of goods which gives the highest value for the contents of the container.

(*25 marks*)

End of Question Paper