The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2012–2013**

Graph Theory **2 hours 30 minutes**

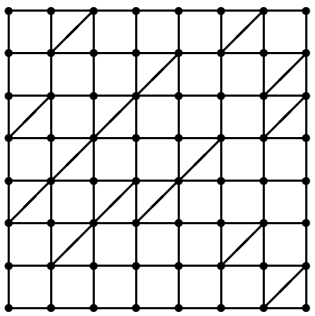
*Answer **four** questions. You are advised **not** to answer more than four questions: if you do, only your best four will be counted.*

1 (i) (a) List all different (i.e. non-isomorphic) trees on 6 vertices. ***(5 marks)***

(b) A compound has chemical formula C_6H_nF. Given that fluorine (F) has valency 1, what value of n will ensure that the molecular model is a tree? ***(5 marks)***

(c) For this value of n, how many structural isomers does it have? Justify your answer. ***(7 marks)***

(ii) By drawing a suitable graph, determine whether the braced framework shown is rigid. If it is rigid, how many bracings must be removed to make it a minimum bracing? If it is not rigid, what is the minimum number of additional braces needed to make it rigid, and how many existing braces would then need to be removed to make it a minimum bracing? Justify all your answers. ***(8 marks)***

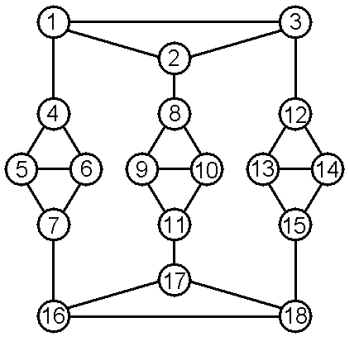


2 (i) Explain why a 3-regular graph must have an even number of vertices. *(2 marks)*

(ii) Prove that if a graph is 3-regular and Hamiltonian then it is 3-edge-colourable. *(6 marks)*

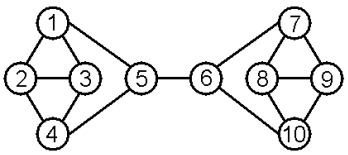
(iii) Give an example (with justification) of a graph which is 4-regular and Hamiltonian but which is not 4-edge-colourable. *(3 marks)*

(iv) What is the chromatic index of the following graph? Is it Hamiltonian? *(7 marks)*



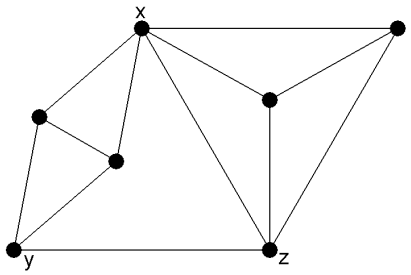
(v) State the four-colour theorem, and give an equivalent statement in terms of edge colouring. (You do not need to prove this equivalence.) *(3 marks)*

(vi) What is the chromatic index of the following graph? Comment on your answer. *(4 marks)*



3 (i) Find the chromatic polynomial of the graph G shown below. You may use standard relations without proof provided they are clearly stated.

(13 marks)



(ii) We wish to colour the vertices of G with the six colours red, green, blue, yellow, cyan and magenta, so that adjacent vertices have different colours. In how many ways can this be done so that

(a) x is red? ***(2 marks)***

(b) y is green and z is blue? ***(3 marks)***

(c) x is yellow and y is cyan? ***(3 marks)***

(d) at least one vertex is magenta? ***(4 marks)***

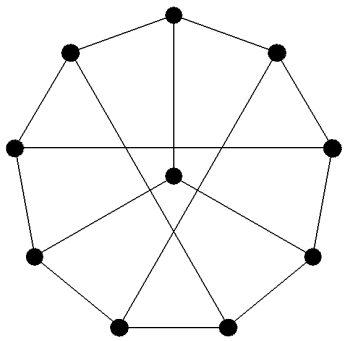
4 (i) (a) State and prove Euler's formula for a plane, connected graph.

(6 marks)

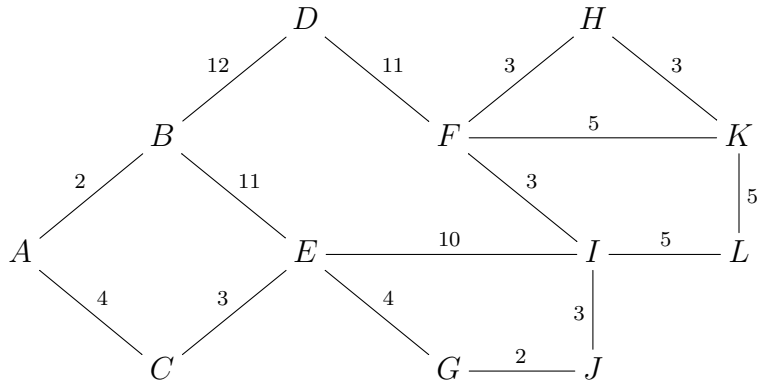
(b) Let G be a simple, connected plane graph in which every vertex has degree at least 3. Suppose that G has 2 faces of degree 3, and every other face has degree at least 5, of which p have degree exactly 5. Prove that $p \geq 6$. ***(10 marks)***

(c) What is the smallest number of edges G may have? Give an example to show that G may have this number of edges, and prove that it cannot have fewer. ***(5 marks)***

(ii) State Kuratowski's theorem and show that the graph below is not planar by finding a suitable subdivision. ***(4 marks)***



5 (i) The graph below shows a network of streets with lengths as marked. A gritter must be driven along a route which covers every street at least once and starts and ends at the same point.



(a) State a result which allows you to relate the minimum possible length of such a route to the length of the shortest path between two specific vertices. Use the shortest path algorithm to find the shortest path between the appropriate vertices. ***(10 marks)***

(b) Describe Fleury's algorithm, and write down an optimal route for the gritter. What is the total length of the route? ***(5 marks)***

(ii) A project consists of eleven tasks. The duration in days of each task and the other tasks which must precede it are given in the table.

Task	Duration	Preceding tasks
A	8	I
B	6	—
C	4	H
D	9	B
E	2	C, D
F	5	E, G
G	7	A, H
H	10	I
I	2	B, J
J	3	—
K	6	C

Use Fulkerson's algorithm to construct an activity network and find the shortest possible time for completion of the project, and the earliest start time for each task. ***(10 marks)***

End of Question Paper