The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2012-2013**

Applicable Analysis **2 hours 30 minutes**

Answer ***four*** *questions. If you answer more than four questions, only your best four will be counted.*

You may use the following results when answering questions on this paper.

Table of Laplace Transforms	
Function	*Laplace Transform*
$t^{\alpha}e^{bt}(\alpha > -1)$	$\dfrac{\Gamma(\alpha+1)}{(s-b)^{\alpha+1}}$
$\sin at$	$\dfrac{a}{s^2+a^2}$
$\cos at$	$\dfrac{s}{s^2+a^2}$
$f(t)e^{bt}$	$F(s-b)$
$f^{(n)}(t)$	$s^nF(s)-\sum_{k=1}^{n} f^{(k-1)}(0)s^{n-k}$
$tf(t)$	$-F'(s)$

1 (i) Define what is meant by the statement that $\int_a^\infty f(x)\,dx$ exists.

(2 marks)

Prove, **from your definition,** each of the following statements:

(a) $\displaystyle\int_0^\infty \frac{1}{(x+1)(x+2)}\,dx$ exists;

(b) $\displaystyle\int_e^\infty \frac{1}{x\ln x}\,dx$ does not exist.

(6 marks)

(ii) State, without proof, the Comparison Test for convergence and divergence of integrals of the form $\int_a^\infty f(x)\,dx$. Your statement should include conditions under which the results are valid.

(4 marks)

Prove each of the following, stating any standard results you need to use:

(a) $\displaystyle\int_e^\infty \frac{(2+\cos x)}{x\ \ln x}\,dx$ diverges;

(b) $\displaystyle\int_0^\infty \frac{1}{1+x\sqrt{x}}\,dx$ converges.

(7 marks)

(iii) Decide whether each of the following integrals converges or diverges and prove your assertions.

(a) $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x}}$;

(b) $\displaystyle\int_0^1 \frac{dx}{(\cos x)\sqrt{1-x}}$.

(6 marks)

2 (i) State, without proof, the theorem concerning differentiation of an integral of the form $\int_a^\infty f(x,y)\ dx$. Your statement should include conditions under which the result holds. ***(4 marks)***

Show that $\int_0^\infty e^{-x^2}\sin(xy)\,dx$ converges for all $y \in \mathbb{R}$.

Let $c > 0$. Prove that the function F defined on $[-c, c]$ by

$$F(y) \quad = \quad \int_0^\infty e^{-x^2}\sin(xy)\,dx \quad (-c \le y \le c)$$

is differentiable on $[-c, c]$ and that

$$2F'(y) \quad = \quad 1 - y\,F(y) \qquad (*)$$

for $-c \le y \le c$. ***(10 marks)***

Deduce that $(*)$ holds for all $y \in \mathbb{R}$. ***(2 marks)***

(ii) Define the Γ function. ***(2 marks)***

Prove that

(a) $\displaystyle\int_0^\infty x^8 e^{-x^2}\ dx \quad = \quad \frac{105\sqrt{\pi}}{32}$;

(b) $\displaystyle\int_0^1 (\ln x)^9\ dx \quad = \quad -(9\,!)$.

(7 marks)

3 Define the Beta function. State, without proof, the relation between the Beta and Gamma functions. ***(3 marks)***

Prove that

$$B(x,y) \;=\; 2\int_0^{\pi/2} \cos^{2x-1}\theta \sin^{2y-1}\theta \, d\theta \qquad (x>0,\ y>0)$$

and

$$B(x,y) \;=\; \int_0^{\infty} \frac{u^{x-1}}{(1+u)^{x+y}}\, du \qquad (x>0,\ y>0).$$

(4 marks)

Prove each of the following, stating any standard results you need to use:

(a) $$\int_0^{\pi/2} \sin^2\theta\, \sqrt{\tan\theta}\, d\theta \;=\; \frac{3\pi\sqrt{2}}{8}\ ;$$

(b) $$\int_0^{\infty} \frac{t^4}{(1+t^4)^2}\, dt \;=\; \frac{\pi\sqrt{2}}{16}\ .$$

(c) the area enclosed by the curve $|x|^6 + |y|^3 = 1$ is

$$\frac{4}{9\sqrt{\pi}}\, \Gamma(\tfrac{1}{6})\, \Gamma(\tfrac{1}{3})$$

(18 marks)

4 (i) Define what is meant by the statement that $\int_0^\infty f(t)e^{-st}\,dt$ has abscissa of convergence c. ***(2 marks)***

Find the abscissa of convergence of $\int_0^\infty te^{-st}\,dt$, giving reasons for your answer.

(4 marks)

(ii) In each of the following cases, find the function continuous on $[0,\infty)$, with the given Laplace transform:

(a) $\dfrac{2}{(s+1)(s+3)} \quad (s > -1)\,;$

(b) $\dfrac{1}{s^2+2s+2} \quad (s > -1)\,;$

(c) $\dfrac{2}{(s+2)(s^2+2s+2)} \quad (s > -1)\,.$

(8 marks)

(iii) Let $b > 0$. Using Beta and Gamma functions show that

$$\int_0^\infty \frac{x^3}{(x^6+b^2)}\,dx \quad = \quad \frac{\pi}{3\sqrt{3}\,b^{2/3}}\,.$$ ***(7 marks)***

By considering

$$\int_0^\infty \sin\left(x^3\,t\right)\,dx\,,$$

prove that

$$\int_0^\infty \sin x^3\,dx \quad = \quad \frac{\pi}{3\sqrt{3}\,\Gamma(\frac{2}{3})}\,.$$ ***(4 marks)***

5 Let $f : [0,\infty) \to \mathbb{R}$ be continuous and suppose that the Laplace transform $F = L(f)$ exists on (c,∞) for some $c \in \mathbb{R}$. State, without proof, the formula giving $L\left(\dfrac{f(t)}{t}\right)$ in terms of F. Your statement should include sufficient conditions to ensure the validity of the formula. ***(2 marks)***

Prove that the Laplace transform of $\dfrac{\cos t - e^{-t}}{t}$ is given by the formula

$$L\left(\frac{\cos t - e^{-t}}{t}\right) = -\tfrac{1}{2}\ln(1+s^2) + \ln(1+s)\ .$$ ***(7 marks)***

Verify that, for $s \neq -1$,

$$\frac{2}{(s^2+1)(s+1)^2} = \frac{1}{(s+1)^2} + \frac{1}{(s+1)} - \frac{s}{s^2+1}$$ ***(1 mark)***

Hence find the solution of the differential equation

$$t\,\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + ty = -\,2te^{-t} \quad (t \geq 0)$$

such that $y(0) = 0$. ***(15 marks)***

End of Question Paper