The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2012–2013**

Knots and Surfaces **2 hours and 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

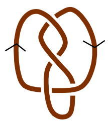
1 (i) Explain carefully the difference between the notions of *knot* and *knot diagram*.

Draw the *Reidemeister moves* and state *Reidemeister's Theorem.*

A link invariant t has integer values, you have four knots K_1, K_2, K_3, K_4. You find that $t(K_1) = 1$, $t(K_2) = 1$, $t(K_3) = 4$, and $t(K_4) = 1$. What can you deduce about the four knots? ***(10 marks)***

(ii) State what it means for a knot to be (a) *reversible* and (b) *amphicheiral.*

An oriented figure eight knot can be represented by the following oriented knot diagram.



Prove that the figure eight knot is both reversible and amphicheiral.

What can you deduce about the Jones polynomial of the figure eight knot from its reversibility and from its amphicheirality? ***(15 marks)***

2 (i) State *Jones' Theorem* on the existence of the Jones polynomial. ***(7 marks)***

(ii) Show that for any link diagram k,

$$f[k \amalg \bigcirc] = -(A^2 + A^{-2})f[k].$$

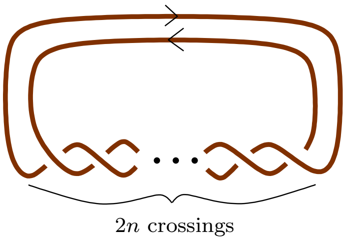
(5 marks)

(iii) Define the oriented link invariant b by evaluating the Jones polynomial at an eighth root of unity: that is, set $\tau := e^{2\pi i/8}$ and $b(k) := f[k](\tau)$. Show that this satisfies

$$b(k_+) - b(k_-) = 2ib(k_0),$$

where k_+, k_- and k_0 are as in the usual Skein Relation. Use result above to evaluate $b(k \amalg \bigcirc)$, for a diagram k. ***(5 marks)***

(iv) Find $b(t_{2n})$ for $n \geq 0$ where t_{2n} is the pictured link diagram. ***(8 marks)***

$t_{2n} :=$ 

3 (i) What is a *compact connected surface*? ***(5 marks)***

(ii) Sketch or describe each of the following subsets of $\mathbb{R}^3$ and decide in each case whether it forms a surface and whether it is compact (no justification is necessary).

(a) $\{(x, y, 0) \in \mathbb{R}^3 : x^2 + y^2 < 1\}$

(b) $\{(x, y, z) \in \mathbb{R}^3 : xyz = 0\}$

(c) $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$

(9 marks)

(iii) Assume that every compact surface can be given a triangulation, that is be decomposed into triangles which intersect either in a whole edge, at a vertex, or not at all. Give the algorithm which proves that every compact connected surface can be formed from a plane model.

Use this algorithm with the tetrahedron triangulation of the sphere to give a surface word representing the sphere. Prove that this word is word equivalent to the standard surface word for the sphere. ***(11 marks)***

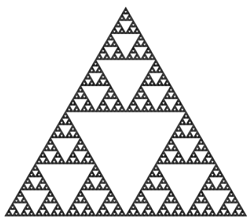
4 (i) State the formula for the Euler characteristic of a surface in terms of a covering pattern of the surface, explaining any symbols you use. ***(3 marks)***

Suppose that you wish to make a polyhedral convex ball from hexagons and pentagons. Show that you need to use exactly 12 pentagons. What is the minimal number of hexagons needed? Justify your answer. ***(8 marks)***

(ii) State the *inclusion-exclusion principle* for the Euler characteristic, and state the Euler characteristic of a *point*, a *line segment* and a *circle*. ***(6 marks)***

If A, B and C are decent subsets of $\mathbb{R}^n$ then use the inclusion-exclusion principle to find an expression for $\chi(A \cup B \cup C)$ in terms of the Euler characteristics of A, B, C and their various intersections. You may assume that the various intersections and unions are also decent.

The Sierpinski gasket is a subset of $\mathbb{R}^2$ which is formed by taking a filled-in triangle, removing the interior of a middle triangle, leaving three triangles round the edge. Interiors of middle triangles are removed from those three triangles leaving nine triangles. This process is repeated indefinitely and what remains is the Sierpinski gasket S. This is *self-similar* in that it contains smaller copies of itself.



Assuming that the Euler characteristic can be defined for S and that it satisfies the inclusion-exclusion principle, calculate $\chi(S)$. Note that you will not get an integer answer. ***(8 marks)***

End of Question Paper