



The  
University  
Of  
Sheffield.

**SCHOOL OF MATHEMATICS AND STATISTICS**

**Autumn 2012-2013**

**Financial Mathematics**

**2 hours and 30 minutes**

*Answer **three** questions. If you answer more than three questions only your best three will be counted.*

**Please leave this exam paper on your desk  
Do not remove it from the hall**

Registration number from U-Card (9 digits)  
to be completed by student

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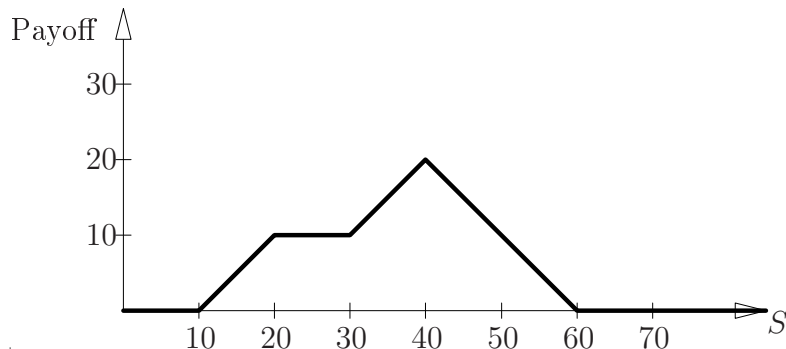
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- 1 (i) Consider the following three bonds with face value of £100:

Time to maturity (in years)	Annual interest (paid every 6 months)	Bond price (in £)
0.5	0	99.25
1.	6%	103.94
1.5	10%	111.00

- (a) Find the 0.5-year spot interest rate. **(1 mark)**
- (b) Use the bootstrap method to find the 1 and 1.5-year spot interest rates. **(8 marks)**
- (c) Suppose that you are offered by a risk free institution the opportunity to deposit or borrow £1000 in one year for a period of six months earning an interest rate of 2.5%. Describe in detail an arbitrage opportunity available to you. **(12 marks)**
- (ii) Consider a twelve-month forward contract on one share of *ABC Plc*. These shares are currently traded for £10 per share, and spot interest rates for all maturities are 2%. Within the next twelve months, *ABC Plc* will pay a single dividend of 50p per share in 6 months.
- (a) Find the present value of the dividend paid in 6 months and the twelve-month forward price of one share of *ABC Plc*. **(2 marks)**
- (b) You are given the opportunity to take a short position in this forward contract at a forward price of £9.75. Describe in detail an arbitrage opportunity available to you. **(10 marks)**

- 2 (i) (a) Describe a portfolio consisting entirely of European call options on the same stock, with same expiration time  $T > 0$ , but with different strike prices, and whose payoff at time  $T$  as a function of  $S$ , the spot price of the stock at time  $T$ , is described by the graph below. *(5 marks)*



- (b) Let  $c_{10}, c_{20}, c_{30}, c_{40}$  and  $c_{60}$  be the prices of the above call options with strike prices 10, 20, 30, 40 and 60, respectively, and let  $p_{60}$  be the price of a European put option on the same stock, with expiration at time  $T$  and with strike price 60. By comparing the payoff of the portfolio in (a) and the payoff of the put option above, describe an inequality involving  $c_{10}, c_{20}, c_{30}, c_{40}, c_{60}$  and  $p_{60}$ . *(5 marks)*
- (ii) Let  $p$  be the price of a European put option on stock which pays no dividends; let  $X$  be the strike price of the option,  $S$  be the spot price of the stock,  $T$  be the time to the option's expiration (in years) and  $r$  be the  $T$ -year spot interest rate. By comparing two portfolios consisting of options, cash and stock, prove that

$$p > Xe^{-rT} - S. \quad (10 \text{ marks})$$

- (iii) The price of a stock which pays no dividends is currently £8. Over each of the next two 1-year periods the stock price will either increase by 50% or decrease by 50%. Suppose that all interest rates are constant and equal to 3%.
- (a) Consider a one-year European call option on this stock with strike price £10. Describe a portfolio consisting of an amount of stock and some number of these call options whose value in one year is known with certainty. *(4 marks)*
- (b) Use a binomial tree to find the price of a two-year American put option on this stock with strike price £9. *(6 marks)*
- (c) Describe all circumstances when a rational investor should exercise the option described in (b) before its expiration. *(3 marks)*

- 3 (i) Explain the principle of risk-neutral valuation. (4 marks)
- (ii) (a) State the mathematical definition of Brownian motion. (5 marks)
- (b) State Ito's Lemma. (3 marks)
- (c) Consider a forward contract on stock paying no dividends maturing at time  $T$  and let  $F(t)$  be its forward price at time  $0 \leq t \leq T$ . Assume all interest rates are constant and equal to  $r$ . Recall that  $F(t) = S(t)e^{r(T-t)}$ , where  $S(t)$  is the spot price of the stock at time  $t$ . Use Ito's Lemma and the fact that

$$dS = \mu S dt + \sigma S dB$$

to find the stochastic process followed by  $F$ . (5 marks)

- (iii) In this question we consider a European call option and a European put option, both with the same underlying stock, the same strike price  $X$  and the same maturity time  $T > 0$ . For any time  $0 \leq t \leq T$  let  $S_t = S$  denote the spot price of the underlying stock, let  $c(S, t)$  be the price of European call and let  $p(S, t)$  be the price of European put option at time  $t$ . Assume that all spot interest rates are constant and equal to  $r$ . Assume also that the underlying stock price follows the Ito process

$$dS = \mu S dt + \sigma S dB$$

- (a) Verify that  $f(S, t) = Xe^{-r(T-t)} - S$  is a solution of the Black-Scholes partial differential equation

$$\frac{\partial f}{\partial t} + rS \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf.$$

(4 marks)

- (b) Deduce that  $g(S, t) = Xe^{-r(T-t)} - S + c(S, t) - p(S, t)$  is also a solution of the Black-Scholes partial differential equation. (4 marks)
- (c) By considering the case  $t = T$ , show that (b) implies that

$$c(S, t) + Xe^{-r(T-t)} = p(S, t) + S. \quad (8 \text{ marks})$$

- 4 (i) Explain the following terms in the context of portfolio theory:
- (a) feasible set, *(2 marks)*
  - (b) efficient frontier, *(2 marks)*
  - (c) the market portfolio, *(2 marks)*
  - (d) the beta coefficient of a stock, and *(2 marks)*
  - (e) the security market line. *(2 marks)*
- (ii) Consider a world where there are only two risky investments: *Greed Plc* and *Safety First Inc.* stocks.

	Number of shares	Price per share	Expected return	Standard deviation of return
Greed Plc	1,000	£20	20%	50%
Safety First Inc.	2,000	£10	5%	10%

The correlation between the returns of these two stocks is 0.25.

- (a) What is the market portfolio? *(3 marks)*
- (b) What are the expected return and standard deviation of returns of the market portfolio? *(4 marks)*
- (c) Find the beta coefficient of *Greed Plc*. *(5 marks)*
- (d) What should be the risk-free return in this world, if one existed? *(5 marks)*
- (e) Assume that risk-free deposits are available. Of all portfolios with expected returns of 8% consisting of a combination of risk-free investments and the two risky investments above, which one has the lowest standard deviation of returns. *(6 marks)*

**End of Question Paper**