The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2012–2013**

MAS377 Mathematical Biology **2 hours**

Marks will be awarded for your best ***four*** *answers.*

1 The dynamics of two interacting populations, N and M, are given by the following ordinary differential equations:

$$\frac{dN}{dt} = N(a - bN + cM)$$
$$\frac{dM}{dt} = M(\alpha - \beta M + \gamma N),$$

where $a, b, c, \alpha, \beta, \gamma$ are positive constants.

(i) Describe, in words, the ecological interaction between N and M. ***(3 marks)***

(ii) Using the given ordinary differential equations, sketch the two qualitatively different phase portraits for this system, the first identifying three (biologically feasible) equilibria, the second identifying four. You should show the nullclines, equilibrium points and qualitative directions of flow. Find the condition on the model parameters that differentiates between the two cases. ***(8 marks)***

(iii) Calculate the Jacobian, J, of the system at the general equilibrium (N_*, M_*). Show that the 'N-only' and 'M-only' equilibria are always saddle points. ***(6 marks)***

(iv) Show that the internal equilibrium found in the four-equilibria case in part (ii) is stable provided $b/c > \gamma/\beta$ (note, you are not required to explicitly state the equilibrium values). Relate this to the condition that differentiated between your two phase portraits in part (ii). ***(8 marks)***

2 The dynamics of a host population exposed to some infectious disease are given by the following ordinary differential equations:

$$\begin{aligned}\frac{dS}{dt} =& \mu(S+I+R) - \beta SI - \mu S \\ \frac{dI}{dt} =& \beta SI - (\mu+\gamma)I \\ \frac{dR}{dt} =& \gamma I - \mu R,\end{aligned}$$

where μ, β, γ are positive constants.

(i) Explain why and how this system can be fully expressed by just two equations. ***(3 marks)***

(ii) Use the substitutions, $X = S/N$, $Y = I/N$, $b = \mu/(\beta N)$, $c = \gamma/(\beta N)$ and $\tau = \beta N t$, where $N = S + I + R$, to fully non-dimensionalise this model. ***(7 marks)***

(iii) Identify the two equilibria of your non-dimensionalised model, and explain what they mean in terms of the long-term persistence of disease in the population. ***(6 marks)***

(iv) Assess the stability of the two equilibria by calculating the Jacobian of the system. Explain how the resulting conditions relate to the quantity $R_0 = \beta N/(\mu + \gamma)$. ***(9 marks)***

3 A model for the regulation of a gene is given by

$$\begin{aligned}\frac{dm}{dt} &= -\mu m + s(t)\\ \frac{dp}{dt} &= m - \mu p,\end{aligned}$$

where m and p represent the concentrations of the mRNA and protein associated with the gene, respectively, and μ is a positive constant.

(i) State the cellular processes that are represented by each term in the model equations. ***(4 marks)***

(ii) If $s = s_0$ for $0 \le t \le T$ and $s = 0$ for $t > T$, determine an expression for $m(t)$ given that $m(0) = 0$. Sketch the time-course for $m(t)$ for $t \ge 0$. ***(8 marks)***

(iii) If $p(0) = 0$, use the expression for $m(t)$ obtained from part (ii) to derive an expression for $p(t)$ for $0 \le t \le T$. Show that $p(t) \approx kt^2$ for small t, and find an expression for k. ***(6 marks)***

(iv) Let

$$m_* = \lim_{T\to\infty} m(T) \quad \text{and} \quad p_* = \lim_{T\to\infty} p(T).$$

Show that the time T_1 such that $m(T_1) = 0.9m_*$ is approximately $T_1 = 3.3\tau$, where τ is the half-life of the mRNA. Show that $p(T_1) \approx \frac{2}{3}p_*$. ***(7 marks)***

4 A model for delayed lateral inhibition between two cells is given by

$$\frac{dX_1}{dt} = -\mu X_1 + g\left(X_2(t-\tau)\right)$$
$$\frac{dX_2}{dt} = -\mu X_2 + g\left(X_1(t-\tau)\right),$$

where $X \geq 0$, $g(X) > 0$ is a monotonically decreasing function, μ is a positive constant, and τ is a non-negative constant delay.

(i) Show that the model always has a uniform steady state solution $X_1 = X_2 = X_*$. Determine the linearised system about the uniform steady state by setting

$$X_i(t) = X_* + x_i(t), \quad i = 1, 2.$$

Show that the variables $y(t) = x_1(t) + x_2(t)$, $z(t) = x_1(t) - x_2(t)$ satisfy the linear equations

$$\frac{dy}{dt} = -\mu y - \gamma y(t-\tau)$$
$$\frac{dz}{dt} = -\mu z + \gamma z(t-\tau),$$

where you should define the parameter γ. ***(7 marks)***

(ii) By considering solutions of the form $y(t) \propto e^{\lambda_1 t}$, $z(t) \propto e^{\lambda_2 t}$, show that

$$\lambda_1 + \mu = -\gamma e^{-\lambda_1 \tau}, \qquad \lambda_2 + \mu = \gamma e^{-\lambda_2 \tau}.$$

Hence show that $z(t)$ grows in time if $\gamma > \mu$, and that if $\gamma - \mu$ is small, then λ_2 can be approximated by

$$\lambda_2 \approx \frac{\gamma - \mu}{1 + \gamma\tau}.$$

(6 marks)

(iii) Show that $\lambda_1 < 0$ when $\tau = 0$, and show graphically that real solutions for λ_1 cease to exist when τ is greater than a critical value (which you do not need to determine). Show that there exists a value $\tau = \tau_H$ such that $\lambda_1 = i\omega$, $\omega \in \mathbb{R}$ if $\gamma > \mu$ and find an expression for ω. ***(7 marks)***

(iv) Show that $\mu \tan(\omega\tau_H) + \omega = 0$ and hence show graphically that the oscillatory period of $y(t)$ when $\tau = \tau_H$ lies between $2\tau_H$ and $4\tau_H$. ***(5 marks)***

5 Consider a spatial model for a harvested fish population

$$\frac{\partial U}{\partial t} = rU\left(1 - \frac{U}{k}\right) - EU + D\frac{\partial^2 U}{\partial x^2}, \quad x \geq 0, \qquad (1)$$

$$\frac{\partial U}{\partial x} = 0 \quad \text{when } x = 0, \qquad (2)$$

where $U(x,t)$ is the fish population level at a distance x from a shoreline, and D, r, k and E are positive constants.

(i) Describe briefly the biological meaning of the terms in the model, and the meaning of the boundary condition at $x = 0$. ***(4 marks)***

(ii) Show that the model has a non-zero *spatially uniform* steady state $U_* > 0$ only if $E < r$. Assuming $E < r$, find U_* and determine whether or not the state $U(x,t) = U_*$ is stable to spatially uniform perturbations.

(6 marks)

(iii) Now assume that fishing is regulated such that $E = E_0 < r$ in the region $0 \leq x < H$. Also assume that $E \gg r$ in the region $x \geq H$, such that we can approximate $U(x,t) = 0$ for $x \geq H$. Linearise the system about the steady state $U = 0$. Considering small perturbations of the form

$$U(x,t) = e^{at}\left(\alpha \cos kx + \beta \sin kx\right),$$

show that the boundary conditions at $x = 0$ and $x = H$ imply that $\beta = 0$ and

$$k = \frac{(n+1)\pi}{2H}, \quad n = 0, 1, 2, \ldots$$

(7 marks)

(iv) Show that spatial perturbations can grow only if

$$H > \frac{\pi}{2}\sqrt{\frac{D}{r - E_0}}$$

and sketch the form of the fastest growing perturbation to the steady state. ***(8 marks)***

End of Question Paper