The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Autumn Semester 2012–2013

Topics in Advanced Fluid Mechanics — **2 hours 30 minutes**

*Marks will be awarded for your best **four** answers.*

1 We consider the 3D Euler equations for an incompressible fluid of a constant unit density.

(1) Derive the Lagrangian form of the equations of motion

$$\frac{Du_i}{Dt}\frac{\partial x_i}{\partial a_j} = -\frac{\partial p}{\partial a_j},$$

where $\dfrac{\partial x_i}{\partial a_j}$, $i, j = 1, 2, 3$ denotes the Jacobian matrix associated with the transformation between spatial coordinates $\boldsymbol{x}$ and material coordinates $\boldsymbol{a}$. ***(4 marks)***

(2) Derive

$$\frac{D}{Dt}\left(u_i \frac{\partial x_i}{\partial a_j}\right) = -\frac{\partial}{\partial a_j}\left(p - \frac{|\boldsymbol{u}|^2}{2}\right).$$

(7 marks)

(3) Hence derive the Weber transform

$$u_i(\boldsymbol{a}, t)\frac{\partial x_i}{\partial a_j} - u_j(\boldsymbol{a}, 0) = -\frac{\partial \psi}{\partial a_j} \qquad (1.1)$$

by suitably defining ψ. ***(7 marks)***

(4) Identify (1.1) as a form of the impulse $\boldsymbol{\gamma} = \boldsymbol{u} + \nabla\phi$, after suitable arrangements. State which gauge you have chosen. ***(7 marks)***

2 We consider Burgers equation in $\mathbb{R}^1$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}, \tag{2.1}$$

with an initial condition $u(x, 0) = u_0(x)$.

(1) Introducing a velocity potential ϕ by $u = \dfrac{\partial \phi}{\partial x}$, derive its equation

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)^2 = \nu \frac{\partial^2 \phi}{\partial x^2}, \tag{2.2}$$

by choosing a constant of integration suitably. ***(7 marks)***

(2) Show that (2.2) is invariant under the following set of transformations

$$x \to \alpha x, \;\; t \to \alpha^2 t,$$

where $\alpha(> 0)$ is an arbitrary parameter. ***(4 marks)***

(3) By assuming $\phi = F(\theta)$, where θ is a solution of the heat diffusion equation

$$\frac{\partial \theta}{\partial t} = \nu \frac{\partial^2 \theta}{\partial x^2},$$

derive the ordinary differential equation

$$\frac{d^2 F(\xi)}{d\xi^2} = \frac{1}{2\nu}\left(\frac{dF(\xi)}{d\xi}\right)^2. \tag{2.3}$$

(7 marks)

(4) Determine ϕ explicitly by solving (2.3). ***(7 marks)***

3 Consider a model equation for vorticity ω defined in $\mathbb{R}^1$:

$$\frac{\partial \omega}{\partial t} = \omega H[\omega], \tag{3.1}$$

with an initial condition

$$\omega(x, t = 0) = \omega_0(x).$$

Here $H[\omega]$ denotes the Hilbert transform on $\mathbb{R}^1$

$$H[\omega](x) = \frac{1}{\pi} \fint_{-\infty}^{\infty} \frac{\omega(y)}{x - y} dy,$$

where $\fint$ is a principal value integral.

You may assume the following formulas

$$H\left[\frac{a}{x^2 + a^2}\right] = \frac{x}{x^2 + a^2}, \quad H\left[\frac{x}{x^2 + a^2}\right] = -\frac{a}{x^2 + a^2},$$

where a is a constant.

(1) Show that a solution of the following form

$$\omega(x, t) = \frac{a(t)}{x^2 + a(t)^2}$$

is inconsistent with (3.1), that is, $a \equiv 0$ is the only possible solution. ***(6 marks)***

(2) Show that a solution of the following form

$$\omega(x, t) = \frac{x}{x^2 + a(t)^2}$$

is consistent with (3.1) by determining $a(t)$ in terms of $a_0 = a(0)$. ***(6 marks)***

(3) Determine $\max_x \omega(t)$ and $\max_x H[\omega](t)$ and the distance between these maxima. Assume that $a(0) < 0$. ***(7 marks)***

(4) Sketch the graphs of $\omega(x, t)$ and $H[\omega](x, t)$ at some $t > 0$. ***(6 marks)***

4 We consider a system of N point-vortices:

$$\kappa_i \frac{dx_i}{dt} = \frac{\partial H}{\partial y_i}, \ \kappa_i \frac{dy_i}{dt} = -\frac{\partial H}{\partial x_i}, \ \text{(no summation)},$$

where (x_i, y_i), $i = 1, 2, \ldots, N$, are coordinates of a point vortex of strength κ_i. Here,

$$H = \frac{1}{4\pi} \sum_{i,j=1}^{N}{}' \kappa_i \kappa_j f(r_{ij}),$$

$r_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$, $f(r)$ is smooth except for $r = 0$ and $\sum{}'$ denotes a summation excluding $j = i$.

(1) Show that the equations of motion can be written as

$$\frac{dx_i}{dt} = \frac{-1}{2\pi} \sum_{j=1}^{N}{}' \kappa_j (y_i - y_j) g(r_{ij}),$$

$$\frac{dy_i}{dt} = \frac{1}{2\pi} \sum_{j=1}^{N}{}' \kappa_j (x_i - x_j) g(r_{ij})$$

Express $g(r)$ in terms of $f(r)$. ***(6 marks)***

(2) Show that

$$\sum_{i=1}^{N} \kappa_i x_i, \ \sum_{i=1}^{N} \kappa_i y_i, \ \sum_{i=1}^{N} \kappa_i (x_i^2 + y_i^2)$$

are constants of motion for general $f(r)$. ***(6 marks)***

(3) Show that H is a constant of motion for general $f(r)$. ***(3 marks)***

(4) Show that the motion of a vortex pair: $N = 2$, $\kappa_1 = -\kappa_2 (\equiv \kappa > 0)$, is a parallel translation with a constant speed, for $f(r) = \log \frac{1}{r}$ (Case 1: the conventional point vortices) and $f(r) = \frac{1}{r}$ (Case 2; a modified version). State in which case the vortices move faster if the mutual distance is sufficiently short. ***(10 marks)***

5 Consider the dynamical equation for a vortex patch in complex notation

$$\frac{\partial z(\alpha,t)}{\partial t} = -\frac{1}{2\pi}\int_0^{2\pi} \log|z(\alpha,t)-z(\beta,t)|\frac{\partial z(\beta,t)}{\partial \beta}d\beta, \qquad (5.1)$$

where $z(\alpha,t)$, $0 \le \alpha \le 2\pi$ denotes the position of the boundary of the patch.

(1) Derive an equation for $\omega(\alpha,t) \equiv \dfrac{\partial z(\alpha,t)}{\partial \alpha}$ as

$$\frac{\partial \omega(\alpha,t)}{\partial t} = -\frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\left(\frac{\omega(\alpha,t)}{z(\alpha,t)-z(\beta,t)}\right)\omega(\beta,t)d\beta,$$

where Re denotes the real part. ***(8 marks)***

(2) Derive

$$\frac{\partial \omega(\alpha,t)}{\partial t} = \frac{i}{2}\omega(\alpha,t) - \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Im}\left(\frac{\omega(\alpha,t)}{z(\alpha,t)-z(\beta,t)}\right)\omega(\beta,t)d\beta, \qquad (5.2).$$

where Im denotes the imaginary part. You may assume the formula

$$\frac{1}{2\pi i}\int_{\partial D}\frac{f(\zeta)d\zeta}{\zeta - z} = \frac{1}{2}f(z), \quad \text{for } z \in \partial D,$$

where f is analytic in domain D on the complex plain and its boundary ∂D.

(8 marks)

(3) Show that the second term on the right-hand-side of the equation (5.2) vanishes for $z = z_0(\alpha) \equiv e^{i\alpha}$. Hence obtain a particular solution

$$z(\alpha,t) = e^{it/2}z_0(\alpha).$$

Give a physical meaning of this solution. ***(9 marks)***

End of Question Paper