The University Of Sheffield.

Data provided: Formula Sheet

SCHOOL OF MATHEMATICS AND STATISTCS **Autumn Semester 2012-13**

MAS 420 Signal Processing **2 hours**

Answer **four** questions. If you answer more than four questions, only your best four will be counted.

1 a) You are given that a periodic signal, $f_T(t)$, with period *T* may be written as a Fourier series in the form

$$f_T(t) = \sum_{n=-\infty}^{\infty} c_n \exp(in\,\sigma t)$$

where *n* is an integer, σ = 2π/*T* is the fundamental frequency, and the complex coefficients c_n are defined as

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f_T(t)\exp(-in\sigma t)dt.$$

If we define

$$f(t) = \begin{cases} f_T(t) & |t| < T/2 \\ 0 & |t| \geq T/2 \end{cases}$$

prove that the complex Fourier series coefficients, c_n, may be expressed as

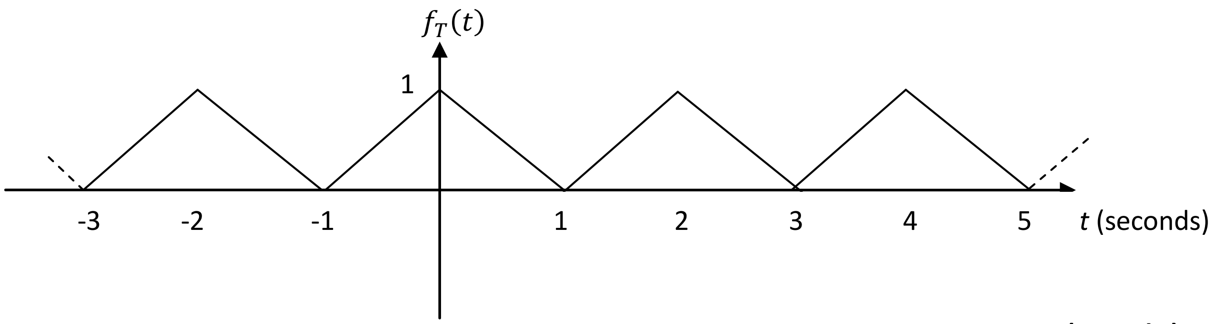
$$c_n = \frac{1}{T}F(n\sigma)$$

where $F(\omega)$ is the Fourier transform of $f(t)$.

(3 marks)

b) Write an expression for the real periodic signal $f_T(t)$ shown below in terms of the triangle function $q_a(t)$.

State the period, T, the fundamental frequency, σ (in rad/s), and the DC level (mean value) of $f_T(t)$, and show that its mean power is 1/3.



(6 marks)

c) Use part (a) to show that the complex Fourier series coefficients, c_n, for the signal $f_T(t)$ shown in the figure are given by

$$c_n = \begin{cases} 1/2 & n = 0 \\ 0 & n \text{ even and } n \neq 0 \\ \dfrac{2}{n^2\pi^2} & n \text{ odd} \end{cases}$$

and express $f_T(t)$ as a sine/cosine series.

(7 marks)

d) Sketch the power spectrum of $f_T(t)$ over the interval [-4π, 4π] rad/s.

(3 marks)

e) Using Parseval's theorem and your answers above, prove that

$$\pi^4 = 96 \sum_{k=1}^{\infty} \frac{1}{(2k-1)^4}.$$

(2 marks)

f) What percentage of the power of the signal $f_T(t)$ would be lost if it was passed through a filter with the transfer function $p_{4\pi}(\omega) - p_{\pi/2}(\omega)$?

(4 marks)

2 a) If $f(t) \leftrightarrow F(\omega)$ is a Fourier transform pair, prove that $F(t) \leftrightarrow 2\pi f(-\omega)$. Use this formula to find the Fourier transform of the signal $\frac{1}{6+it}$.

(4 marks)

b) The convolution of two finite-energy signals *f(t)* and *g(t)* is defined as

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(s)g(t-s)ds.$$

Without using Fourier transforms, show that the convolution of f(t) and g(t) is commutative and linear.

(4 marks)

c) Consider the signal

$$f(t) = \left(\frac{1}{a}\right) p_a(t) * p_b(t) \qquad \text{where } b \geq a > 0.$$

Making use of clear sketches and without using Fourier transforms, calculate $f(t)$. Sketch the function $f(t)$ and show that it may be expressed as

$$f(t) = \left(\frac{b}{a} + 1\right) q_{(a+b)}(t) - \left(\frac{b}{a} - 1\right) q_{(b-a)}(t).$$

(7 marks)

d) Use the convolution theorem and the formula sheet to verify the result in part (c) when $a = b$.

(3 marks)

e) Find the energy of the signal $f(t)$ defined in part (c) when $a = 1$ and $b = 2$.

(3 marks)

f) Using the result in part (c) and the convolution theorem, prove the relation

$$\sin(x)\sin(3x) = \sin^2(2x) - \sin^2(x).$$

(4 marks)

3 a) For a function $f(t)$ that is continuous at $t = 0$, show that

$$f(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) d\omega,$$

where $F(\omega)$ is the Fourier transform of $f(t)$.

(2 marks)

b) Make a sketch of $|f(t)|^2$ for a signal $f(t)$ with finite energy, E, and for which $|f(0)|^2 \geq |f(t)|^2$. Using your diagram, explain the concept and significance of the equivalent rectangle resolution,

$$\tau = \frac{E}{|f(0)|^2}.$$

(3 marks)

c) Prove that if $f(t)$ has finite energy and is Ω-band-limited, then $\tau\Omega \geq \pi$, with equality if and only if the signal's Fourier transform, $F(\omega)$, is a scalar multiple of $p_\Omega(\omega)$. What is the corresponding signal in the time domain?

(5 marks)

d) Consider the signal $f(t) = 3\text{sinc}^2(2t)$. Calculate and sketch the signal spectrum, $F(\omega)$, and find

(i) its bandwidth, Ω (rad/s),

(ii) its energy, and

(iii) the equivalent rectangle resolution, τ.

Verify that for this signal $\tau\Omega > \pi$.

(7 marks)

e) The signal is passed through a low-pass filter with system transfer function $H(\omega) = p_3(\omega)$ to produce the signal $g(t)$.

(i) Show that filtering the signal reduces its energy by a fraction 1/64.

(ii) Use the relation in part (a) to find $g(0)$ and hence show that the equivalent rectangle resolution of $g(t)$ is exactly 12% greater than that of $f(t)$.

(iii) Calculate the time-bandwidth product of the filtered signal.

(8 marks)

4 a) Find the Fourier transform, $F(\omega)$, of the signal $f(t) = e^{-3t}U(t)$ and calculate and sketch its amplitude spectrum and phase spectrum.

(6 marks)

b) Prove that if a linear shift-invariant system has a real impulse response $h(t)$ with corresponding system transfer function $H(\omega)$, then $H(\omega) = H^*(-\omega)$. Hence show that the output signal from the system given an input signal $f(t) = \sin(\omega_0 t)$ is

$$\Im\{H(\omega_0)\exp(i\omega_0 t)\}.$$

Similarly show that the system's response to the input signal $f(t) = \cos(\omega_0 t)$ is

$$\Re\{H(\omega_0)\exp(i\omega_0 t)\}.$$

(Note: $\Re\{z\}$ denotes the real part of a complex variable z and $\Im\{z\}$ is its imaginary part.)

(6 marks)

c) Given an input signal $f(t)$, a system outputs the signal

$$g(t) = \int_{t-T/2}^{t+T/2} f(s)ds.$$

Show that the system transfer function, $H(\omega)$, is $T\mathrm{sinc}(\omega T/2)$.

(3 marks)

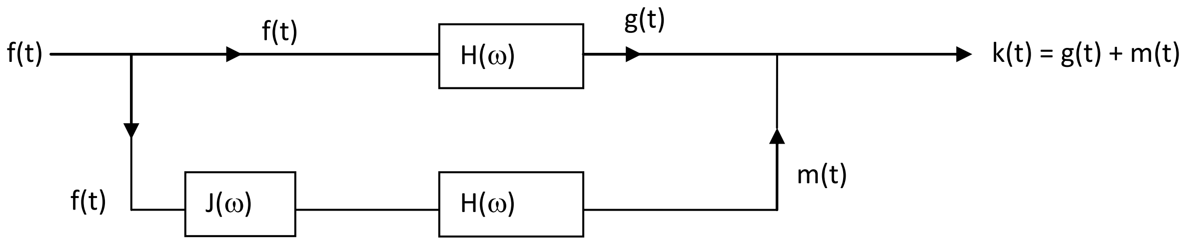
d) Find the response of the system in part (c) to the following input signals:

(i) $f(t) = \delta(t)$;

(ii) $f(t) = 3\sin(2t) + \cos(3t)$.

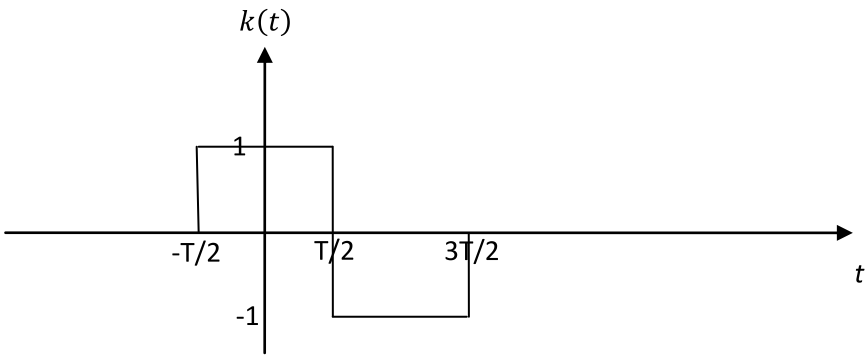
(5 marks)

e) Consider the system shown below, in which the input signal $f(t)$ is split into two branches containing components with system transfer function $H(\omega) = T\mathrm{sinc}(\omega T/2)$ (the integrator in part (c)) and another component with STF $J(\omega) = \alpha\exp(-i\omega t_0)$. The outputs from each branch are summed to give the total output signal $k(t)$.



(i) Write down the STF for the complete system shown.

(ii) By considering the system's impulse response, find the input signal $f(t)$ and the values of α and t_0 resulting in the output signal illustrated below.



(iii) Sketch an alternative system design which has the same STF but uses one fewer components.

(5 marks)

5 a) An analogue signal $f(t)$ is sampled at intervals of T seconds. Assuming that the sampled signal, $f_s(t)$, may be written

$$f_s(t) = f(t)\bar{\delta}_T(t)$$

where

$$\bar{\delta}_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

and $\bar{\delta}_T(t)$ has the Fourier transform $\sigma\bar{\delta}_\sigma(\omega)$, where $\sigma = 2\pi/T$, use the product theorem to show that the Fourier transform of the sampled signal is given by

$$F_S(\omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} F(\omega - n\sigma)$$

where $F(\omega)$ is the Fourier transform of $f(t)$.

If $f(t)$ is Ω-band-limited and $T = 0.1$ seconds, what is the largest value of Ω (in rad/s) for which the spectrum $F_S(\omega)$ consists of non-overlapping copies of $F(\omega)$?

(4 marks)

b) For an Ω-band-limited signal, $f(t)$, define the Nyquist frequency. Explain how the signal may be reconstructed exactly from the sampled signal $f_s(t)$ using a low-pass filter if sampled at a rate greater than the Nyquist frequency.

Show that this process is equivalent to convolving the sampled signal with a sinc function in the time domain.

(5 marks)

c) The analogue signal

$$f(t) = \text{sinc}(4t) + \frac{1}{2}\text{sinc}^2(2t)$$

is sampled every T seconds to give the digital signal $f_s(t)$.

(i) Calculate the Fourier transform of $f(t)$ and sketch its amplitude spectrum.

(ii) Determine the bandwidth of $f(t)$ and its Nyquist frequency.

(5 marks)

d) Sketch a few periods of the amplitude spectrum $|F_s(\omega)|$ of the sampled signal in part (c), when

(i) $T = \pi/6$;

(ii) $T = \pi/3$.

In each case state whether the analogue signal $f(t)$ may be reconstructed exactly from its digital samples using a low-pass filter.

(6 marks)

e) The signal $f(t)$ (defined in part (c)) is sampled at half the Nyquist frequency to produce the samples $f_s(t)$. Another signal $g(t)$ is then reconstructed from $f_s(t)$ by sinc interpolation. Calculate $g(t)$.

(5 marks)

End of Question Paper

Function Definitions:
Rectangular pulse:

$$p_a(t) = \begin{cases} 1 & |t| \le a \\ 0 & |t| > a \end{cases}$$

Triangular pulse:

$$q_a(t) = \begin{cases} 1 - |t|/a & |t| \le a \\ 0 & |t| > a \end{cases}$$

Step function:

$$U(t) = \begin{cases} 1 & t \ge 0 \\ 0 & t < 0 \end{cases}$$

Fourier Transform Pairs: (Note: $\mathrm{sinc}(x) = \sin(x)/x$)

$$\begin{aligned}
p_a(t) &\longleftrightarrow 2a\,\mathrm{sinc}(a\omega) \\
q_a(t) &\longleftrightarrow a\,\mathrm{sinc}^2(a\omega/2) \\
\mathrm{sinc}(at) &\longleftrightarrow \frac{\pi}{a}p_a(\omega) \\
\mathrm{sinc}^2(at) &\longleftrightarrow \frac{\pi}{a}q_{2a}(\omega) \\
\mathrm{e}^{-at}U(t) &\longleftrightarrow \frac{1}{a+i\omega} \\
\delta(t) &\longleftrightarrow 1 \\
\delta(t-t_0) &\longleftrightarrow \mathrm{e}^{-i\omega t_0} \\
1 &\longleftrightarrow 2\pi\delta(\omega) \\
\mathrm{e}^{i\omega_0 t} &\longleftrightarrow 2\pi\delta(\omega-\omega_0) \\
\mathrm{e}^{-t^2/2\sigma^2} &\longleftrightarrow \sigma\sqrt{2\pi}\mathrm{e}^{-\sigma^2\omega^2/2}
\end{aligned}$$

Fourier transform:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\mathrm{e}^{-i\omega t}\,\mathrm{d}t$$

Inverse Fourier transform:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\mathrm{e}^{i\omega t}\,\mathrm{d}\omega$$

Duality theorem: If $f(t) \longleftrightarrow F(\omega)$ then $F(t) \longleftrightarrow 2\pi f(-\omega)$

Scaling: If $f(t) \longleftrightarrow F(\omega)$ then $f(at) \longleftrightarrow \frac{1}{|a|}F(\omega/a)$.

Translation: If $f(t) \longleftrightarrow F(\omega)$ then $f(t-t_0) \longleftrightarrow \mathrm{e}^{-i\omega t_0}F(\omega)$.

Frequency Shift: If $f(t) \longleftrightarrow F(\omega)$ then $\mathrm{e}^{i\omega_0 t}f(t) \longleftrightarrow F(\omega-\omega_0)$

Fourier Series: If $f_T(t)$ is periodic with period T then, with $\sigma = 2\pi/T$, the complex Fourier series is

$$f_T(t) = \sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{in\sigma t}$$

where

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f_T(t)\mathrm{e}^{-in\sigma t}\ \mathrm{d}t$$

Likewise, the real Fourier series is

$$f_T(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\sigma t + b_n \sin n\sigma t)$$

where

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f_T(t)\cos n\sigma t\ \mathrm{d}t$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f_T(t)\sin n\sigma t\ \mathrm{d}t$$

Parseval's Theorem: If V is a Hilbert space, $\{\phi_n\}$ is an orthonormal basis for V and $f = \sum_n c_n\phi_n$, then

$$\|f\|^2 = \sum_{n=-\infty}^{\infty} |c_n|^2$$

Plancherel's Theorem: If $f(t) \longleftrightarrow F(\omega)$ and $g(t) \longleftrightarrow G(\omega)$ then

$$\int_{-\infty}^{\infty} f(t)g^*(t)\ \mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)G^*(\omega)\ \mathrm{d}\omega$$

Energy Theorem: If $f(t) \longleftrightarrow F(\omega)$ then

$$\int_{-\infty}^{\infty} |f(t)|^2\ \mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2\ \mathrm{d}\omega$$

Convolution Theorem: If $f(t) \longleftrightarrow F(\omega)$ and $g(t) \longleftrightarrow G(\omega)$ then

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(s)g(t-s)\ \mathrm{d}s \longleftrightarrow F(\omega)G(\omega)$$

Product Theorem: If $f(t) \longleftrightarrow F(\omega)$ and $g(t) \longleftrightarrow G(\omega)$ then

$$f(t)g(t) \longleftrightarrow \frac{1}{2\pi}F(\omega) * G(\omega).$$

[fsheet.tex: 31st October 2012, Rev. 1.1]