Data provided: Formula Sheet

MAS422

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2012–2013**

MAS422 Magnetohydrodynamics **2 hours**

Answer all four questions.

1 (i) Using Ohm's law

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}),$$

show, for the case where the conductivity σ is not constant, that

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B} - \nabla \eta \times (\nabla \times \mathbf{B})$$

where $\eta = \dfrac{1}{\mu_0 \sigma}$. ***(6 marks)***

(ii) If the vector potential $\mathbf{A} = (0, xe^{-z}, 0)$, find the magnetic field $\mathbf{B}$ and sketch the field lines clearly indicating the direction of the field. ***(9 marks)***

(iii) Consider a rotating object to be symmetric around a rotation axis. Thus, using cylindrical coordinates (r, ϕ, z)

$$\mathbf{v} = r\Omega(r, z)\hat{\phi}$$

is independent of ϕ. Here, Ω is the angular velocity. Now, consider that the object has axisymmetric poloidal field, frozen into plasma. Show that a steady state is possible only if Ω is constant along field lines.
(Hint: use $\mathbf{B} = \nabla \times \dfrac{1}{r}\psi(r, z)\hat{\phi}$). ***(10 marks)***

2 (i) Consider magnetic field in cylindrical polar coordinates (r, ϕ, z). If a magnetic field $\mathbf{B} = \mathbf{B}(r)$ varies with r alone, why can it not possess a radial component (B_r)? ***(3 marks)***

 Question 2 continued on next page

2 (continued)

(ii) Consider a coronal loop with $\mathbf{B}_0 = 10$ Gauss (10^{-3} Tesla), $L = 5\times10^7$ m and $\rho_0 = 8\times10^{-13}$ kg m^{-3}. For these values the Alfvén speed is approximately $v_A = B_0/\sqrt{\mu\rho_0} = 10^6$ m s^{-1} and the wave number is $k = 2\pi/L = 1.3\times10^{-7}$ m^{-1}. Find the period of standing wave oscillation in the coronal loop.

(3 marks)

(iii) For a compressible fluid, the equation of continuity is given by

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0.$$

Using the standard vector identity

$$\nabla\times(\mathbf{F}\times\mathbf{G}) = \mathbf{F}(\nabla\cdot\mathbf{G}) - \mathbf{G}(\nabla\cdot\mathbf{F}) + (\mathbf{G}\cdot\nabla)\mathbf{F} - (\mathbf{F}\cdot\nabla)\mathbf{G},$$

show that the induction equation, in the case where the magnetic diffusivity $\eta = 0$, can be rewritten as

$$\frac{\partial}{\partial t}\left(\frac{\mathbf{B}}{\rho}\right) + (\mathbf{u}\cdot\nabla)\frac{\mathbf{B}}{\rho} = \left(\frac{\mathbf{B}}{\rho}\cdot\nabla\right)\mathbf{u}$$

(9 marks)

(iv) Show that a force-free field obeys the equation

$$\nabla\times\mathbf{B} = \lambda\mathbf{B},$$

where λ is constant along each field line. ***(1 mark)***

For a one-dimensional force-free field of the form

$$\mathbf{B} = B_y(x)\hat{\mathbf{y}} + B_z(x)\hat{\mathbf{z}},$$

show that

$$B_y^2 + B_z^2 = B_0^2,$$

where B_0 is constant. ***(6 marks)***

In the particular case when $B_y = B_0\cos x$, find $B_z(x)$ and λ. ***(3 marks)***

3 (i) Calculate $(\mathbf{B}\cdot\nabla)\dfrac{\mathbf{B}}{\mu}$ for the magnetic field given by

$$\mathbf{B} = y\hat{x} + x\hat{y}.$$

What do you expect the directions of the magnetic tension to be in the x-axis? Show them with the arrows after plotting the fieldlines. ***(6 marks)***

 Question 3 continued on next page

3 (continued)

(ii) For $B(x,t) = \phi(t)e^{-x^2/(4\eta t)}$ to satisfy

$$\frac{\partial B}{\partial t} = \eta \frac{\partial^2 B}{\partial x^2},$$

what is the condition on $\frac{\partial \phi}{\partial t}$? ***(6 marks)***

(iii) Consider a horizontal magnetic field $B(z)\hat{\mathbf{x}}$ in equilibrium with a plasma, satisfying

$$O = -\frac{d}{dz}\left(p + \frac{B^2}{2\mu}\right) - \rho g$$

where $\rho = p/RT$.

If $T = T_0$ and $B(z) = B_0 \frac{z}{H}$, where T_0, B_0, $H(= \frac{RT}{g})$ are constants, show that

$$\frac{dp}{dz} + \frac{p}{H} = -cz$$

where $c = \frac{B_0^2}{\mu H^2}$.

Solve this equation for $p(z)$ if $p(0) = p_0$ in terms of H, p_0, $\beta = \frac{2\mu p_0}{B_0^2}$.

(9 marks)

(iv) Consider a star with $R_* = 10^{11}$ cm and $B_* \approx 100$G. If this star collapses to a neutron star with radius $R_{NS} = 10^6$ cm, estimate the neutron star's magnetic field strength noting that the magnetic flux is conserved.

(4 marks)

4 (i) Ignoring viscosity, gravity and diffusivity, write down closed MHD equations using the material derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$$

(4 marks)

(ii) Assume the background fluid to be stationary and homogeneous with constant density ρ_0 and pressure p_0 as a function of position. Further, consider a constant background magnetic field of strength B_0, that points in the z-direction, write down the linearised MHD equations (linearising about the background quantities). ***(4 marks)***

(iii) Seeking the plane wave solutions of the form $\sim \exp\,(i\mathbf{k}\cdot\mathbf{x} - i\omega t)$ for the perturbed quantities, find the dispersion relations for Alfvén waves and magneto-acoustic waves using the linearised MHD equations obtained in (ii). ***(17 marks)***

Formulae Sheet

$$\nabla^2 \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times \nabla \times \mathbf{A}$$

	u	v	w	f	g	h
cartesian	x	y	z	1	1	1
spherical	r	θ	ϕ	1	r	$r \sin\theta$
cylindrical	r	ϕ	z	1	r	1

$$\nabla \cdot \mathbf{V} = \frac{1}{fgh}\left[\frac{\partial}{\partial u}(ghV_u) + \frac{\partial}{\partial v}(fhV_v) + \frac{\partial}{\partial w}(fgV_w)\right]$$

$$\nabla \times \mathbf{V} = \frac{1}{gh}\left[\frac{\partial}{\partial v}(hV_w) - \frac{\partial}{\partial w}(gV_v)\right]\hat{u} \;+\; \frac{1}{fh}\left[\frac{\partial}{\partial w}(fV_u) - \frac{\partial}{\partial u}(hV_w)\right]\hat{v} + \frac{1}{fg}\left[\frac{\partial}{\partial u}(gV_v) - \frac{\partial}{\partial v}(fV_u)\right]\hat{w}$$

End of Question Paper