



The
University
Of
Sheffield.

MAS424

SCHOOL OF MATHEMATICS AND STATISTICS

**Spring Semester
2012–2013**

**Differential Equations: Case Studies in Applied
Mathematics**

2 hours

Attempt all questions.

- 1 (i) In the Canadian wilds, mountain bears eat berries and catch salmon in the rivers, and the salmon eat various insects which live near the water surface. A simple model for the bear and salmon populations, which assumes that each population exists in a finite environment, is given by the following equations for x and y :

$$\dot{x} = x(a_0 - b_0x + c_0y), \quad \dot{y} = y(a_1 - b_1y - c_1x);$$

in which the six parameters $(a_0, b_0, c_0, a_1, b_1, c_1)$ are strictly positive. Find all the critical points of this model and classify those three for which $xy = 0$, taking care to distinguish the separate cases $a_1b_0 > a_0c_1$ and $a_1b_0 < a_0c_1$. *Note:* You need only write down the equations satisfied for the critical point satisfying $xy \neq 0$. Do not solve them. (10 marks)

- (ii) Find the general solution of the autonomous system described by the equations:

$$\begin{aligned} \frac{dx}{dt} &= x + 2y; \\ \frac{dy}{dt} &= 3x + 2y. \end{aligned}$$

State the nature of the critical point. (5 marks)

- (iii) The Fisher equation is a model for the spread of an advantageous gene through a population. It can be written in the form

$$\frac{\partial u}{\partial t} = Au \left(1 - \frac{u}{K}\right) + D \frac{\partial^2 u}{\partial x^2},$$

where u represents the population density, A and D are positive constants, and K is the carrying capacity.

- (a) By rescaling the Fisher equation using

$$U = \frac{u}{K}, \quad T = At, \quad X = x \left(\frac{A}{D}\right)^{\frac{1}{2}},$$

show that

$$\frac{\partial U}{\partial T} = U(1 - U) + \frac{\partial^2 U}{\partial X^2}. \tag{†}$$

- (b) By assuming a wave solution of the form $U(X - cT)$, convert equation (†) into a 2nd order ordinary differential equation with independent variable $z = X - cT$.
- (c) Use the substitution $V = U'$ to convert the 2nd order ordinary differential equation into a pair of 1st order ordinary differential equations. Hence find the co-ordinates of the critical points of this system.

(10 marks)

- 2 (i) Suppose that $x^*(t)$ is a minimising curve for the functional

$$J[x] = \int_{t_0}^{t_1} f(x, \dot{x}, t) dt$$

where $x(t_0)$ and $x(t_1)$ are fixed. Assuming, for simplicity, that the admissible curves belong to \mathcal{C}_2 , then derive the first necessary condition

$$\frac{\partial f}{\partial x} - \frac{d}{dt} \left(\frac{\partial f}{\partial \dot{x}} \right) = 0 \quad (*)$$

that the function $x^*(t)$ must satisfy. You may assume the result that if $g(t)$ is continuous in $[a, b]$ then $\int_a^b \eta(t)g(t)dt = 0$ for all $\eta(t)$ implies that $g(t) = 0$ for all $t \in [a, b]$. **(8 marks)**

- (ii) If t does not appear explicitly in the functional so that $f \equiv f(x, \dot{x})$, then the Euler-Lagrange equation (*) given in part (a) has an explicit first integral given by

$$f - \dot{x} \frac{\partial f}{\partial \dot{x}} = \text{constant}.$$

Prove that this latter result is, in fact, a first integral of the Euler-Lagrange equations. **(5 marks)**

- (iii) Show that the function $x^*(t) = \frac{t}{2}$ is an extremal for the problem

$$J[x] = \int_0^2 (1 - \dot{x})^2 (1 + \dot{x})^2 dt, \quad x(0) = 0, \quad x(2) = 1,$$

but that it is *not* a minimising curve for the problem. **(12 marks)**

3 Consider the system of equations

$$\begin{aligned} \dot{x} &= -y - xy^2(x^2 + y^2 - 1), \\ \dot{y} &= x - y^3(x^2 + y^2 - 1). \end{aligned} \quad (*)$$

- (i) Show that the coordinate origin is a critical point of this system, and it is a centre. (5 marks)
- (ii) Use Theorem 1 from the Formula Sheet to show that system (*) does not have periodic solutions in the region $y > 1$. (5 marks)
- (iii) Use the variable substitution

$$x = r \cos \theta, \quad y = r \sin \theta,$$

to obtain the system of equations for r and θ ,

$$\dot{r} = -r^3(r^2 - 1) \sin^2 \theta, \quad \dot{\theta} = 1. \quad (\dagger)$$

Use this result to show that system (*) has a limit cycle solution defined by $x^2 + y^2 = 1$. (6 marks)

- (iv) Linearize system (\dagger) near the limit cycle solution $r = 1$. Solve this linearized system for arbitrary initial conditions. Use this solution to show that the limit cycle is asymptotically stable. (You may assume the trigonometric identity $2 \sin^2 \theta = 1 - \cos 2\theta$). (9 marks)

- 4
- (i) (a) State Kepler's three laws of planetary motion. (3 marks)
 - (b) Show that for circular orbits, angular momentum conservation is equivalent to Kepler's 2nd law. (4 marks)
 - (c) Explain what is meant by the 'Principle of Equivalence'. (2 marks)

- (ii) The radial and transverse equations for the motion of a planet, P , about its sun, at location O , can be written, in the usual notation, as:

$$\begin{aligned} \ddot{R} - R\dot{\theta}^2 &= -\frac{k}{R^2}; \\ R^2\dot{\theta} &= h. \end{aligned}$$

You are given that when $\theta = 0$, the planet is moving in a direction that is instantaneously perpendicular to OP . By making the substitution $U = \frac{1}{R}$, derive the solution to this equation in the form

$$\frac{l}{R} = e \cos \theta + 1,$$

where $l = \frac{h^2}{k}$ and e are constants. (12 marks)

Describe the paths of the orbits for different ranges of the value of e . (4 marks)

End of Question Paper

List of Basic Formulae and Theorems

Theorem 1: If a periodic solution of the system of equations

$$\dot{x} = f(x, y), \quad \dot{y} = g(x, y)$$

exists in a simply connected region, then $f_x + g_y = 0$ somewhere in that region.

Corollary: There are no periodic solutions in any simply connected region where $f_x + g_y \neq 0$ everywhere.

Theorem 2: The orbit \mathcal{C} of a periodic solution must enclose at least one critical point.

Orthogonality conditions for trig functions

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = 0, \quad \int_{-\pi}^{\pi} \cos mx \cos nx \, dx = 0 \quad \text{when } m \neq n.$$

$$\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0.$$

Extremals of functional

$$J[y] = \int_{x_0}^{x_1} f(y, y', x) \, dx$$

are the solutions to the Euler-Lagrange equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) = 0.$$