The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2012–13**

Analytic Number Theory **2 hours 30 minutes**

*Answer **Question 1** and three other questions. You are advised **not** to answer more than three of the questions 2 to 5: if you do, only your best three will be counted.*

Please leave this exam paper on your desk
Do not remove it from the hall

Registration number from U-Card (9 digits)
to be completed by student

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1 (i) Given a character χ of the group $(\mathbb{Z}/N\mathbb{Z})^\times$, explain how one defines the associated *Dirichlet L-function* $L(s,\chi)$, and indicate its region of convergence. Describe, without proof, the behaviour of $L(1,\chi)$ and $\lim_{\sigma\to 1+} \sum \frac{\chi(p)}{p^\sigma}$.

(7 marks)

(ii) This question asks you to illustrate the proof of Dirichlet's Theorem in a specific case.

(a) Verify that $(\mathbb{Z}/10\mathbb{Z})^\times$ is a cyclic group. List the characters of $(\mathbb{Z}/10\mathbb{Z})^\times$, indicating which are the non-trivial characters.

(6 marks)

(b) Prove that $L(1,\chi) \neq 0$ for each non-trivial character χ on your list.

(5 marks)

(c) Prove that there are infinitely many primes congruent to $7 \pmod{10}$.

(7 marks)

2 (i) Let $\Phi(X)$ be the polynomial $X^8 + 1$.

(a) Show that the set of primes p for which there is an integer n with $p|\Phi(n)$ is infinite. ***(4 marks)***

(b) Now let a be an integer, and let p be an odd prime dividing $\Phi(a)$. Show that p divides $a^{16} - 1$, and that the order of a in $(\mathbb{Z}/p\mathbb{Z})^\times$, the unit group of $\mathbb{Z}/p\mathbb{Z}$, is 16. ***(6 marks)***

(c) Deduce that there are infinitely many primes of the form $16k + 1$.

(4 marks)

(ii) State the Prime Number Theorem. ***(1 mark)***

Let $0 < a < b$. Evaluate

$$\lim_{x\to\infty} \frac{\pi(ax)}{\pi(bx)}.$$

Deduce that if x is sufficiently large then the interval $[ax, bx]$ contains at least 2013 distinct primes. ***(10 marks)***

3 (i) Let n be a positive integer, and let p be a prime.

(a) Write down a formula for the highest power of p that divides $n!$, and calculate the number of zeros at the end of 2013!. ***(4 marks)***

(b) Show that $\lfloor x+y \rfloor - \lfloor x \rfloor - \lfloor y \rfloor = 0$ or 1 for all $x, y \in \mathbb{R}$. (Recall that $\lfloor x \rfloor$ is the greatest integer not exceeding x.) ***(3 marks)***

Hence show that if p^r divides $\binom{2n}{n}$ then $p^r \leq 2n$. ***(4 marks)***

(c) Prove that if $n \geq 3$ and $2n/3 < p \leq n$, then $\binom{2n}{n}$ is not divisible by p. ***(4 marks)***

(ii) State Bertrand's Postulate. ***(1 mark)***

Show, using Bertrand's Postulate or otherwise, that

(a) there are infinitely many primes that begin with the digit 1; ***(2 marks)***

(b) the sum

$$\frac{1}{m} + \frac{1}{m+1} + \ldots + \frac{1}{m+n}$$

is never an integer for $1 \leq m \leq n, \ m, n \in \mathbb{Z}$. ***(7 marks)***

[Hint: Consider the largest prime not exceeding $m+n$.]

4 Let $f, g, h : \mathbb{N} \longrightarrow \mathbb{C}$ be three arithmetic functions related by the following formal product of Dirichlet series:

$$\left(\sum_{n=1}^{\infty} \frac{f(n)}{n^s}\right)\left(\sum_{n=1}^{\infty} \frac{g(n)}{n^s}\right) = \sum_{n=1}^{\infty} \frac{h(n)}{n^s}. \qquad (\star)$$

(i) Write down the relation between f, g and h. Write down the (formal) Euler product expansion of $\sum \frac{f(n)}{n^s}$ if f is a multiplicative arithmetic function, and give a simplified form when f is totally multiplicative. ***(4 marks)***

Now let a be a real number and let

$\sigma_a(n) :=$ the sum of the a-th powers of the divisors of n.

(For example, $\sigma_a(4) = 1 + 2^a + 4^a$.) As usual, $d(n)$ is the number of divisors of n.

(ii) By choosing appropriate f and g in $(\star)$, derive an expression for $\sum_{n=1}^{\infty} \frac{\sigma_a(n)}{n^s}$ in terms of the Riemann zeta function. ***(4 marks)***

Also, write down the Euler product expansion of the Riemann zeta function $\zeta(s)$. ***(1 mark)***

(iii) Derive a formula for $d\big(p^k\big)$ if p is a prime number and $k \geq 1$. ***(2 marks)***

(iv) Establish the identity

$$1 + (1+\alpha)x + (1+\alpha+\alpha^2)x^2 + (1+\alpha+\alpha^2+\alpha^3)x^3 + \ldots$$
$$= \frac{1}{(1-x)(1-\alpha x)},$$

and derive from it the following identity:

$$1 + 2(1+\alpha)x + 3(1+\alpha+\alpha^2)x^2 + 4(1+\alpha+\alpha^2+\alpha^3)x^3 + \ldots$$
$$= \frac{1-\alpha x^2}{(1-x)^2(1-\alpha x)^2}.$$

(8 marks)

(v) Show that

$$\sum_{n=1}^{\infty} \frac{\sigma_a(n)d(n)}{n^s} = \frac{\zeta(s)^2\zeta(s-a)^2}{\zeta(2s-a)}.$$

You may assume that $\sigma_a(n)$, $d(n)$ are multiplicative. ***(6 marks)***

5 (i) Recall that $B_n(x)$, the n-th Bernoulli polynomial, and B_n, the n-th Bernoulli number, are defined by the generating series

$$\frac{te^{xt}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \qquad \text{and} \qquad \frac{t}{e^t-1} = \sum_{n=0}^{\infty} B_n\frac{t^n}{n!}.$$

(a) Show that $B_n(x+1) - B_n(x) = nx^{n-1}$ for all $n \geq 1$. Deduce that $B_n(1) = B_n$ for all $n \geq 2$. ***(5 marks)***

(b) Show that $B_n(1-x) = (-1)^n B_n(x)$ for all $n \geq 0$. Hence or otherwise show that $B_n = 0$ if $n \geq 3$ is an odd number. ***(6 marks)***

(c) Let $m \geq 1$ be an integer. Show that

$$\frac{te^{xt}}{e^{t/m}-1} = \sum_{n=0}^{\infty} \frac{B_n(mx)}{m^{n-1}}\frac{t^n}{n!},$$

and that

$$B_n(mx) = m^{n-1}\sum_{k=0}^{m-1} B_n\Big(x + \frac{k}{m}\Big).$$

(7 marks)

(ii) Let $f : (0,\ \infty) \longrightarrow \mathbb{C}$ be a continuously differentiable function. Show that

$$f(1)+f(2)+\ldots+f(n) = \frac{1}{2}(f(1)+f(n)) + \int_1^n f(x)\,dx + \int_1^n f'(x)P_1(x)\,dx.$$

Here $P_1(x) := x - \lfloor x \rfloor - \dfrac{1}{2}$ is the first periodic Bernoulli function.

(7 marks)

End of Question Paper