The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Spring Semester 2012–2013

MAS436: Functional Analysis — 2 hours 30 minutes

Answer ***four*** *questions. If you answer more than four questions, only your best four will be counted.*

Throughout this paper, unless otherwise stated, all normed vector spaces are either over the field of real numbers, $\mathbb{R}$, or the field of complex numbers, $\mathbb{C}$

1 (i) Let V be a normed vector space over the field $\mathbb{R}$. Say what is meant by a bounded linear map $f\colon V \to \mathbb{R}$, and define the norm of such a map. Prove that this norm satisfies the axioms required to turn the dual space $V^* = Hom(V, \mathbb{R})$ into a normed vector space. ***(9 marks)***

(ii) Prove that the above dual space V^* is complete. ***(7 marks)***

(iii) State the Hahn-Banach theorem. ***(3 marks)***

(iv) Prove that the linear map $\tau\colon V \to (V^*)^*$ defined by the formula

$$\tau(v)(f) = f(v) \qquad f \in V^*,\ v \in V$$

is an isometry. ***(6 marks)***

2 (i) Let V and W be normed vector spaces. Say what is meant by the statement that a linear map $T: V \to W$ is *bounded*, and prove that if T is a bounded linear map, then the kernel, $\ker T$, is closed. ***(4 marks)***

(ii) Let $C^\infty[0,1]$ be the vector space of all real-valued infinitely-differentiable functions $f: [0,1] \to \mathbb{R}$, equipped with the norm

$$\|f\| = \sup\{|f(t)| \mid t \in [0,1]\}.$$

Let $C_0^\infty[0,1]$ be the subspace of all infinitely differentiable functions $f: [0,1] \to \mathbb{R}$ such that $f(0) = 0$.

(a) Prove that we have a bounded linear map $T: C^\infty[0,1] \to C_0^\infty[0,1]$ defined by the formula

$$(Tf)(x) = \int_0^x f(t)\, dt.$$

(4 marks)

(b) Prove that the above map T is bijective. ***(3 marks)***

(c) Does the map T have a *continuous* inverse ? Justify your answer. ***(4 marks)***

(iii) (a) Say what is meant by an open map, and state the open mapping theorem. ***(3 marks)***

(b) Prove that $C_0^\infty[0,1]$ is a closed subspace of the space $C^\infty[0,1]$? ***(4 marks)***

(c) Is the normed vector space $C^\infty[0,1]$ complete? Justify your answer. ***(3 marks)***

3 (i) (a) State Zorn's lemma, including definition of the terms *maximal* and *upper bound*. ***(4 marks)***

(b) Define what is meant by an orthonormal basis for a Hilbert space H. ***(3 marks)***

(c) Prove that every Hilbert space has an orthonormal basis. ***(7 marks)***

(ii) (a) For a function $f\colon \mathbb{R} \to \mathbb{C}$, define $Jf\colon (-\pi, \pi) \to \mathbb{C}$ by

$$Jf(x) = \frac{1}{\sqrt{2}}(\tan(x/2) + i)f(\tan(x/2)).$$

Show that the above formula describes a unitary operator $J\colon L^2(\mathbb{R}) \to L^2(-\pi, \pi)$. ***(7 marks)***

(b) Use the fact that the functions

$$e_k(x) = \frac{1}{\sqrt{2\pi}} \exp(ikx) \qquad k \in \mathbb{Z}$$

form an orthonormal basis of $L^2(-\pi, \pi)$ to prove that the functions

$$f_n(x) = \frac{(i - x)^n}{\sqrt{\pi}(i + x)^{n+1}}$$

form an orthonormal basis of $L^2(\mathbb{R})$. ***(4 marks)***

4 (i) Let H be a Hilbert space. State what is meant by the statements that a map $T\colon H \to H$ is a *self-adjoint operator*, and that a map $U\colon H \to H$ is a *unitary operator*. ***(4 marks)***

(ii) Let $A\colon H \to H$ be a bounded linear operator. Show that we have a well-defined operator

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} \qquad (A^0 = I).$$

(6 marks)

(iii) Let $n \in \mathbb{N}^{\geq 0}$. Show that $(e^A)^n = e^{nA}$. ***(6 marks)***

(iv) Show that the operator e^A is invertible (even if A is not invertible), and $(e^A)^{-1} = e^{-A}$. ***(5 marks)***

(v) Let $A\colon H \to H$ be self-adjoint. Show that e^{iA} is unitary. ***(4 marks)***

5 (i) (a) Let A be a unital Banach algebra, and let $x \in A$ be an element such that $\|x\| < 1$. Prove that the element $1 - x$ is invertible. ***(8 marks)***

(b) Define the *spectrum* of an element $x \in A$. Prove this it is a bounded subset of $\mathbb{C}$. ***(5 marks)***

(c) State the spectral mapping theorem for polynomials. ***(2 marks)***

(ii) Let A be the Banach algebra of all bounded linear maps $\ell^1 \to \ell^1$. Determine the spectrum of the following operators from ℓ^1 to ℓ^1.

(a) $S(a_1, a_2, a_3, \ldots) = (a_2, a_3, \ldots)$.

(b) $T(a_1, a_2, a_3, \ldots) = (a_3 - 2a_1, a_4 - 2a_2, a_5 - 2a_3, \ldots)$.

[Hint: For part (a), find the eigenvalues of S. You may use without proof the fact that the spectrum of S is closed.]

(10 marks)

6 (i) Define what is meant by the statement that a linear map between normed vector spaces is a *compact operator*. ***(2 marks)***

(ii) Let $K: V \to W$ be a compact operator between normed vector spaces V and W. Let (x_n) be a bounded sequence in V. Prove that (Kx_n) has a convergent subsequence. ***(4 marks)***

(iii) Prove that any bounded linear map with finite-dimensional image is compact. ***(4 marks)***

(iv) Prove that the operator $S: \ell^2 \to \ell^2$ defined by the formula

$$S(a_1, a_2, a_3, \ldots) = (a_1, \frac{a_2}{2}, \frac{a_3}{3}, \ldots)$$

is compact.

[You may use without proof here the fact that a norm-limit of a sequence of compact operators is again compact] ***(5 marks)***

(v) What is the definition of a Fredholm operator? ***(2 marks)***

(vi) Define $T: \ell^2 \to \ell^2$ by the formula

$$T(a_1, a_2, a_3, \ldots) = (a_1 + a_3, \frac{a_2}{2} + a_4, \frac{a_3}{3} + a_5, \ldots)$$

Show that T is Fredholm, and calculate $Index(T)$. You may use any standard results from the theory of Fredholm operators to do this. ***(8 marks)***

End of Question Paper