The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2012–2013**

Optics and Symplectic Geometry **2 hours 30 minutes**

*Answer **four** questions. You are advised **not** to answer more than four questions: if you do, only your best four will be counted.*

Throughout the paper I denotes an identity matrix and J denotes a matrix of the form $\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$. All matrices have real entries. The standard symplectic form Ω on $\mathbb{R}^{2n}$ is defined by $\Omega(Z, Z') = Q \cdot P' - P \cdot Q'$, where $Z = (Q, P)$ and $Z' = (Q', P')$ are elements of $\mathbb{R}^{2n}$.

In Questions 2 to 5 you may, if you wish, use results from Question 1.

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Do not remove it from the hall

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1 (i) (a) Define what it means for a $2n \times 2n$ matrix S to be symplectic.

(b) Prove that the $2n \times 2n$ matrix J is invertible. **(5 marks)**

(ii) Let $S = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ be a $2n \times 2n$ matrix in block form, where A, B, C and D denote $n \times n$ matrices.

(a) Prove that S is symplectic if and only if the three equations

$$A^{\mathsf{T}}C = C^{\mathsf{T}}A, \qquad B^{\mathsf{T}}D = D^{\mathsf{T}}B, \qquad A^{\mathsf{T}}D - C^{\mathsf{T}}B = I,$$

hold.

(b) Assume that S is symplectic. Using only the results of (i) and (a), and basic facts from linear algebra, show that S is invertible and establish a formula for S^{-1} in block form. **(10 marks)**

(iii) Now let $S = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ be a symplectic $2n \times 2n$ matrix and assume that $\det A \neq 0$.

(a) Show that $T := \begin{bmatrix} I_n & 0 \\ -CA^{-1} & I_n \end{bmatrix}$ is symplectic.

(b) Calculate TS, simplifying your answer.

(c) Hence, or otherwise, show that

$$\det S = \det A \, \det(D - CA^{-1}B).$$

(10 marks)

2 (i) Show that every $S \in Sp(4)$ has determinant $+1$. You may use, if you wish, the following result:

> *Let $\Theta\colon (\mathbb{R}^4)^4 \to \mathbb{R}$ be a multilinear, skew-symmetric form such that $\Theta(e_1, e_2, e_3, e_4) = 1$, where e_1, e_2, e_3, e_4 is the standard basis of $\mathbb{R}^4$. Then, for any $Z_1, Z_2, Z_3, Z_4 \in \mathbb{R}^4$, $\Theta(Z_1, Z_2, Z_3, Z_4)$ is the determinant of the matrix with columns Z_1, Z_2, Z_3, Z_4.*

(15 marks)

(ii) Figure 1 shows the refraction of a light ray across a parabolic boundary. Show, using neat diagrams if you wish, that $\varphi = \theta + \psi - \frac{\pi}{2}$ and $\varphi' = \theta' + \psi - \frac{\pi}{2}$, where φ' is the angle from the refracted ray to the positive z-axis.

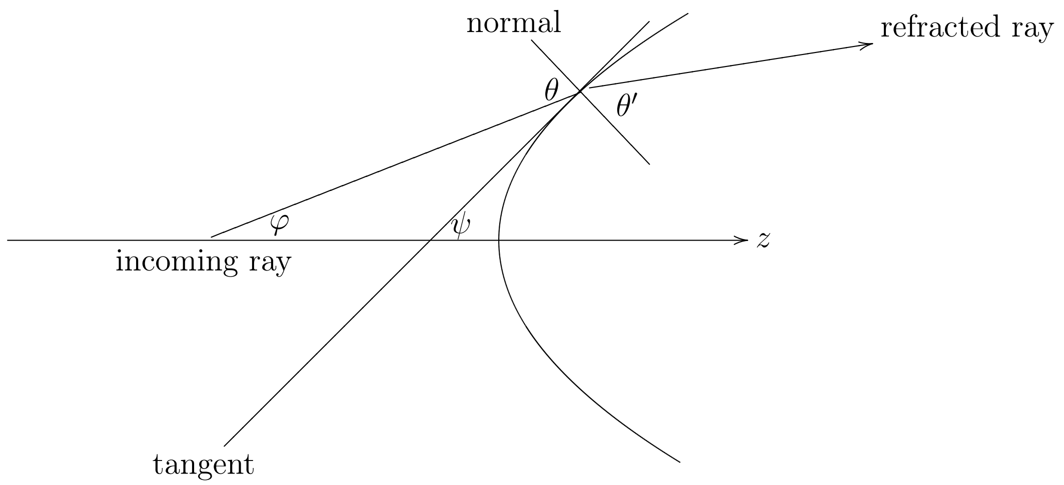

Figure 1: For Question 2(ii)

(10 marks)

3 (a) List the three properties which a map $\omega\colon \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ must have in order to be a symplectic form on $\mathbb{R}^{2n}$. **(3 marks)**

(b) Define the notion of symplectic basis for $(\mathbb{R}^{2n}, \omega)$, where ω is any symplectic form on $\mathbb{R}^{2n}$. **(2 marks)**

(c) Define the symplectic perp $W^\wedge$ of a subspace $W \subseteq \mathbb{R}^{2n}$. **(2 marks)**

(d) Assuming that $n \geqslant 1$, show that there are vectors E_1, F_1 in $\mathbb{R}^{2n}$, such that $\omega(E_1, F_1) = 1$. Deduce that E_1 and F_1 are linearly independent.

(4 marks)

(e) Write $W = \{qE_1 + pF_1 \mid q, p \in \mathbb{R}\}$. Show that $\mathbb{R}^{2n} = W \oplus W^\wedge$ and that $W^\wedge$ is a symplectic subspace of $\mathbb{R}^{2n}$. **(9 marks)**

(f) Using induction, or otherwise, show that $\mathbb{R}^{2n}$ has a symplectic basis.

(5 marks)

4 Denote the unit sphere, centre origin, in $\mathbb{R}^3$ by U. For $X \in U$ write $t_X(U)$ for the tangent plane to U at X and write

$$T(U) = \{(X, Y) \in U \times \mathbb{R}^3 \mid Y \in t_X(U)\}.$$

For each $(X, Y) \in T(U)$ write $L_{(X,Y)}$ for the oriented line which passes through Y, is normal to $t_X(U)$, and has the same orientation as the line from the origin to X.

(a) Draw a neat sketch showing U, a point $X \in U$, the tangent plane $t_X(U)$, and an oriented line $L_{(X,Y)}$. **(5 marks)**

(b) Show that every oriented line in $\mathbb{R}^3$ is $L_{(X,Y)}$ for a unique $(X, Y) \in T(U)$. **(6 marks)**

(c) Show that $T(U)$ can be identified with

$$\{(X, Y) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid X \cdot X = 1,\ Y \cdot X = 1\}. \qquad (*)$$

Now let $(X(t), Y(t))$ be a (smooth) curve in $T(U)$. Using $(*)$ or otherwise, show that

$$\dot{X}(0) \cdot X(0) = 0, \qquad Y(0) \cdot \dot{X}(0) + \dot{Y}(0) \cdot X(0) = 0.$$

where $\dot{X}(t)$ and $\dot{Y}(t)$ denote the derivatives. **(4 marks)**

(d) Take $(X_0, Y_0) \in T(U)$. Let W denote the subspace of $\mathbb{R}^3 \times \mathbb{R}^3$ given by

$$\{(\xi, \eta) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \xi \cdot X_0 = 0,\ \xi \cdot Y_0 + X_0 \cdot \eta = 0\}.$$

Show that W is a symplectic subspace of $\mathbb{R}^6$ with respect to Ω. **(10 marks)**

5 In this question each $\mathbb{R}^{2n}$ has the standard symplectic form Ω.

(a) Let W be a vector subspace of $\mathbb{R}^{2n}$. Define what it means for W to be a Lagrangian subspace of $\mathbb{R}^{2n}$. Define what it means for two Lagrangian subspaces of $\mathbb{R}^{2n}$ to be transversal. **(3 marks)**

(b) Let L be a Lagrangian subspace. Show that $\mathbb{R}^{2n} = L \oplus J(L)$, stating clearly (but not proving) any result from general linear algebra which you use. **(8 marks)**

(c) Let L and L' be Lagrangian subspaces both of which are transversal to both $\mathbb{R}^n \times 0$ and $0 \times \mathbb{R}^n$. State (without proof) the theorem which gives criteria for the existence of $S \in Sp(2n)$ such that $S(\mathbb{R}^n \times 0) = \mathbb{R}^n \times 0$, $S(0 \times \mathbb{R}^n) = 0 \times \mathbb{R}^n$, and $S(L) = L'$. **(4 marks)**

(d) Verify that the following two subspaces of $\mathbb{R}^4$ are Lagrangian.

$$L = \text{span}\{(3, 4, 0, 1),\ (-1, -1, 1, -1)\}, \quad L' = \text{span}\{(4, 6, 1, 1),\ (1, 3, 1, 0)\}.$$

(4 marks)

(e) For L and L' in (d), determine whether or not there exists $S \in Sp(4)$ such that $S(\mathbb{R}^2 \times 0) = \mathbb{R}^2 \times 0$, $S(0 \times \mathbb{R}^2) = 0 \times \mathbb{R}^2$, and $S(L) = L'$. **(6 marks)**

End of Question Paper