The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2012-2013**

Algebra I **2 hours 30 minutes**

*Answer **two** questions of the questions 1 to 3 **and** answer **two** questions of the questions 4 to 6. You are advised **not** to answer more than two questions in each set: if you do, only your best two in that set will be counted.*

1 (i) Let K be a subfield of a field L. Give a definition of $[L : K]$. ***(2 marks)***

(ii) Express the complex number $\dfrac{(1-2i)(1+2i)}{1-i}$ in the form $a+bi$ where $a, b \in \mathbb{R}$. ***(2 marks)***

(iii) Let K and L be fields such that $[K : \mathbb{Q}] = 5$ and $[L : \mathbb{Q}] = 4$. Is it possible that $K \subseteq L$? Justify your answer. ***(4 marks)***

(iv) For each of the subsets J_1, J_2 of $\mathbb{C}$ specified below determine, with justification, whether it is a subfield of $\mathbb{C}$.

(a) $J_1 = \{a + b\sqrt{3} : \ a, b \in \mathbb{Q}\}$, ***(5 marks)***

(b) $J_2 = \{a + b\sqrt{3} + ci : \ a, b, c \in \mathbb{Q}\}$ ***(3 marks)***

(v) Show that $\sqrt{2} \notin J_1$, where J_1 is the set from (iv). Use this to find the subfield of $\mathbb{C}$ generated by the numbers $\{\sqrt{2}, \sqrt{3}\}$ and give a possible $\mathbb{Q}$-basis. Justify your answer. ***(9 marks)***

2 (i) State Gauss's Lemma. ***(3 marks)***

(ii) Prove Gauss's Lemma. ***(8 marks)***

(iii) Give the definition of a simple field extension. ***(2 marks)***

(iv) Let $K = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. Prove that $K = \mathbb{Q}(b)$ where $b = \sqrt{2} + 2\sqrt{3}$. ***(8 marks)***

(v) Find a quartic polynomial over $\mathbb{Q}$ which has b as a root. ***(4 marks)***

3 (i) Let B be a set of (at least 2) points in the plane $\mathbb{R}^2$, and $P, Q \in \mathbb{R}^2$. Explain what it means that the point P is **constructible in one step from** B and the point Q is **constructible from** B. ***(4 marks)***

(ii) Define the set of constructible points. ***(2 marks)***

(iii) Let $(a, b) \in \mathbb{R}^2$. Give a criterion for constructibility of the point (a, b) (via quadratic fields). ***(3 marks)***

(iv) Prove the criterion. You may use the following facts:

- If $(a, b) \in \mathbb{R}^2$ is constructible in one step from B, then $[\mathbb{Q}(B)(a, b) : \mathbb{Q}(B)]$ equals 1 or 2. $\mathbb{Q}(B)$ here denotes the subfield of $\mathbb{R}^2$ generated by the coordinates of points in B.
- If $X \subseteq \mathbb{R}^2$ is the subfield of constructible points, $a \in X$ and $a \geq 0$, then $\sqrt{a} \in X$.

(8 marks)

(v) Using **only** the criterion, show that the point $(\sqrt{2 + \sqrt[4]{3}}, 0)$ is a constructible point. ***(4 marks)***

(vi) Is the point $(\frac{1}{2}, \sqrt[3]{2})$ constructible? Justify your response. ***(4 marks)***

4 Consider the real numbers $\alpha = \sqrt{4 + \sqrt{7}}$, $\beta = \sqrt{4 - \sqrt{7}}$ and let L be the extension $\mathbb{Q}(\alpha)$ of $\mathbb{Q}$.

(i) Determine a quartic polynomial P_α over $\mathbb{Q}$ such that $P_\alpha(\alpha) = 0$. In the following, you may assume that P_α is irreducible. ***(2 marks)***

(ii) Show that $\pm\alpha$ and $\pm\beta$ are roots of P_α. Find expressions for $\alpha \cdot \beta$, $(\alpha + \beta)^2$ and $(\alpha - \beta)^2$. ***(4 marks)***

(iii) Show that β is an element of L and write β as a $\mathbb{Q}$-linear combination of the basis $\{1, \alpha, \alpha^2, \alpha^3\}$ of L. ***(6 marks)***

(iv) Define for a general extension $K \subseteq M$ what it means to be a Galois extension. Show that the extension $\mathbb{Q} \subseteq L$ is a Galois extension and determine the structure of the Galois group $Gal(L|\mathbb{Q})$. ***(7 marks)***

(v) Find all intermediate extensions M with $\mathbb{Q} \subseteq M \subseteq L$. Justify your answer. ***(6 marks)***

5 Let D_4 be the group with elements $\{e, \sigma, \sigma^2, \sigma^3, \tau, \tau\sigma, \tau\sigma^2, \tau\sigma^3\}$ and the composition rules $\tau^2 = e$, $\sigma^4 = e$, $\tau\sigma = \sigma^3\tau$.

(i) Determine the order of all elements of D_4 (you should find five involutions, i.e. elements $g \in D_4 \setminus \{e\}$ with $g^2 = e$). ***(3 marks)***

(ii) You may assume that D_4 has precisely 10 subgroups, precisely three of which have order 4. Give a complete list of these subgroups. For each subgroup H, you should list the elements of H and give the name of a standard group that is isomorphic to H. ***(6 marks)***

(iii) Which of the subgroups of D_4 is normal? ***(5 marks)***

(iv) Give a detailed statement (without proof) of the Galois correspondence for finite extensions $K \subseteq L$. Your answer should contain information about orders of subgroups, degrees and orders of intermediate field extensions, conjugacy and containment between subgroups, and normality of field extensions. ***(5 marks)***

(v) Now assume $Gal(L|K) \cong D_4$. What can you deduce about the intermediate extensions M with $\mathbb{Q} \subseteq M \subseteq L$ and their various containment relations? You should prove your answer by referring to known theorems. ***(6 marks)***

6 Let $L = \mathbb{Q}(i, \sqrt{3}, \sqrt{11})$.

(i) Give a basis for L over $\mathbb{Q}$. You need not prove that your answer is correct. ***(2 marks)***

(ii) Define the Galois group $Gal(M|K)$ of a general extension $K \subseteq M$. Determine the elements of $Gal(L|\mathbb{Q})$ and show that $|Gal(L|\mathbb{Q})| = [L : \mathbb{Q}]$. Deduce that $\mathbb{Q} \subseteq L$ is a Galois extension and give an explicit isomorphism $Gal(L|\mathbb{Q}) \cong \mathbb{Z}_2^3$. ***(8 marks)***

(iii) Determine the subgroups of $Gal(L|\mathbb{Q})$ that correspond, via the Galois correspondence, to the subfields

$$K_1 = \mathbb{Q}(\sqrt{3} + \sqrt{11}),\ K_2 = \mathbb{Q}(i, \sqrt{33}),\ K_3 = \mathbb{Q}(i\sqrt{33}).$$

Are all of these fields Galois extensions of $\mathbb{Q}$? ***(7 marks)***

(iv) How many intermediate fields M with $\mathbb{Q} \subseteq M \subseteq L$ exist such that $[M : \mathbb{Q}] = 4$? Justify your answer. ***(4 marks)***

(v) Show that, if $P \in \mathbb{Q}[X]$ is any irreducible polynomial of odd degree, then P has no root in L. ***(4 marks)***

End of Question Paper