Data provided: Formula Sheet for Section A

MAS6360

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2012–2013**

Geometry I **2 hours 30 minutes**

Answer two questions from Section A and two questions from Section B. If you answer more than two questions from a section, only your best two in that section will be counted.

A list of formulae for Section A is provided on the last page.

Throughout Section B, I denotes an identity matrix and J denotes a matrix of the form $\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$. All matrices have real entries. The standard symplectic form Ω on $\mathbb{R}^{2n}$ is defined by $\Omega(Z, Z') = Q \cdot P' - P \cdot Q'$, where $Z = (Q, P)$ and $Z' = (Q', P')$ are elements of $\mathbb{R}^{2n}$.

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Do not remove it from the hall

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Section A

A1 (i) Consider the map $\mathbf{x} : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$\mathbf{x}(u,v) = \left(u - \frac{u^3}{3} + uv^2\,,\ v - \frac{v^3}{3} + u^2 v\,,\ u^2 - v^2 \right).$$

Let S be the image of $\mathbf{x}$, which is a surface in $\mathbb{R}^3$.

(a) Show that $\mathbf{x}$ is a conformal map between $\mathbb{R}^2$ and S. **(9 marks)**

(b) For each real number c, let γ_c be the curve on S defined by $\gamma_c(t) = \mathbf{x}(t, ct)$ for $t \in \mathbb{R}$. Write down, with an explanation, a curve on S that intersects with every γ_c at a right angle. **(7 marks)**

(ii) Let $a > b > 0$. Consider the ellipse on $\mathbb{R}^2$ defined by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,.$$

Find the maximum and minimum values of its curvature. **(9 marks)**

A2 (i) Let $R > r > 0$. Consider the torus parametrised by

$$\mathbf{x}(\varphi, \theta) = \big((R + r\cos\varphi)\cos\theta, (R + r\cos\varphi)\sin\theta, r\sin\varphi\big)$$

whose first and second fundamental forms are respectively

$$r^2 d\varphi^2 + (R + r\cos\varphi)^2 d\theta^2, \qquad r\, d\varphi^2 + (R + r\cos\varphi)\cos\varphi\, d\theta^2\,.$$

(a) Find the principal curvatures as functions of φ and θ. **(4 marks)**

(b) Write down a curve on the torus whose tangent vectors are always principal. **(3 marks)**

(ii) Let $\mathbf{x}(u,v)$ be a parametrisation of a surface whose mean curvature function is $H(u,v)$ and whose Gaussian curvature function is $K(u,v)$. Given a positive number c, what are the mean and Gaussian curvature functions of the surface parametrised by $c\mathbf{x}(u,v)$? Explain your answer. **(8 marks)**

(iii) Suppose a surface has a parametrisation whose second fundamental form is zero everywhere. Prove that the surface must be (part of) a plane. **(10 marks)**

A3 (i) Consider the cylinder parametrised by $(\cos u, \sin u, v)$ and the curve defined by $\gamma(t) = (\cos t, \sin t, 2t)$. Show that γ is a geodesic on the cylinder.

(7 marks)

(ii) The curve $x = \cosh z$ on the xz-plane is rotated about the z-axis to produce a surface of revolution (a catenoid). This surface can be parametrized by

$$\mathbf{x}(u, \theta) = (\cosh u \cos\theta, \cosh u \sin\theta, u)$$

whose first and second fundamental forms are respectively

$$\cosh^2 u(du^2 + d\theta^2), \qquad -du^2 + d\theta^2.$$

(a) Find the Gaussian curvature as a function of u and θ. **(4 marks)**

(b) For each real number c, the latitude defined by $\gamma_c(t) = \mathbf{x}(c, t)$ is known to have constant geodesic curvature. Using the Gauss-Bonnet formula, show that its value is $\sinh c$. **(14 marks)**

Section B

B1 (i) Show that every $S \in Sp(4)$ has determinant $+1$. You may use, if you wish, the following result:

Let $\Theta\colon (\mathbb{R}^4)^4 \to \mathbb{R}$ be a multilinear, skew-symmetric form such that $\Theta(e_1, e_2, e_3, e_4) = 1$, where e_1, e_2, e_3, e_4 is the standard basis of $\mathbb{R}^4$. Then, for any $Z_1, Z_2, Z_3, Z_4 \in \mathbb{R}^4$, $\Theta(Z_1, Z_2, Z_3, Z_4)$ is the determinant of the matrix with columns Z_1, Z_2, Z_3, Z_4.

(15 marks)

(ii) Figure 1 shows the refraction of a light ray across a parabolic boundary.

Show, using neat diagrams if you wish, that $\varphi = \theta + \psi - \frac{\pi}{2}$ and $\varphi' = \theta' + \psi - \frac{\pi}{2}$, where φ' is the angle from the refracted ray to the positive z-axis.

(10 marks)

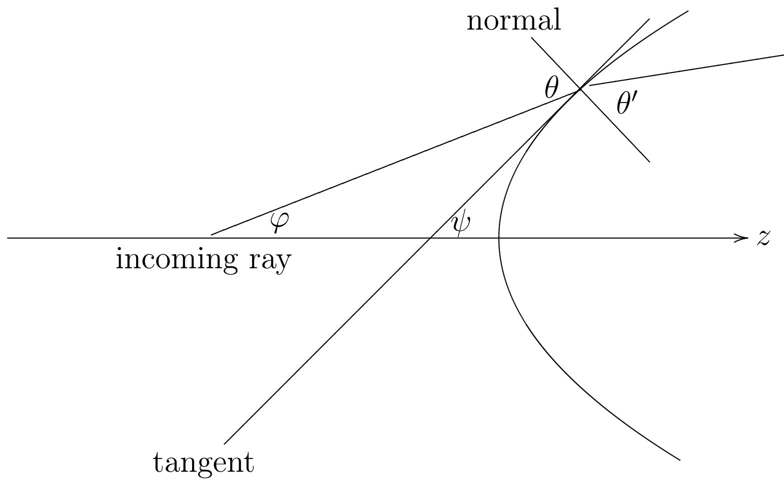

Figure 1: For Question **B1**(ii)

B2 (a) List the three properties which a map $\omega\colon \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ must have in order to be a symplectic form on $\mathbb{R}^{2n}$. **(3 marks)**

(b) Define the notion of symplectic basis for $(\mathbb{R}^{2n}, \omega)$, where ω is any symplectic form on $\mathbb{R}^{2n}$. **(2 marks)**

(c) Define the symplectic perp $W^{\wedge}$ of a subspace $W \subseteq \mathbb{R}^{2n}$. **(2 marks)**

(d) Assuming that $n \geqslant 1$, show that there are vectors E_1, F_1 in $\mathbb{R}^{2n}$, such that $\omega(E_1, F_1) = 1$. Deduce that E_1 and F_1 are linearly independent. **(4 marks)**

(e) Write $W = \{qE_1 + pF_1 \mid q, p \in \mathbb{R}\}$. Show that $\mathbb{R}^{2n} = W \oplus W^{\wedge}$ and that $W^{\wedge}$ is a symplectic subspace of $\mathbb{R}^{2n}$. **(9 marks)**

(f) Using induction, or otherwise, show that $\mathbb{R}^{2n}$ has a symplectic basis. **(5 marks)**

B3 In this question each $\mathbb{R}^{2n}$ has the standard symplectic form Ω.

(a) Let W be a vector subspace of $\mathbb{R}^{2n}$. Define what it means for W to be a Lagrangian subspace of $\mathbb{R}^{2n}$. Define what it means for two Lagrangian subspaces of $\mathbb{R}^{2n}$ to be transversal. **(3 marks)**

(b) Let L be a Lagrangian subspace. Show that $\mathbb{R}^{2n} = L \oplus J(L)$, stating clearly (but not proving) any result from general linear algebra which you use. **(8 marks)**

(c) Let L and L' be Lagrangian subspaces both of which are transversal to both $\mathbb{R}^n \times 0$ and $0 \times \mathbb{R}^n$. State (without proof) the theorem which gives criteria for the existence of $S \in Sp(2n)$ such that $S(\mathbb{R}^n \times 0) = \mathbb{R}^n \times 0$, $S(0 \times \mathbb{R}^n) = 0 \times \mathbb{R}^n$, and $S(L) = L'$. **(4 marks)**

(d) Verify that the following two subspaces of $\mathbb{R}^4$ are Lagrangian.

$$L = \text{span}\{(3,4,0,1),\ (-1,-1,1,-1)\}, \quad L' = \text{span}\{(4,6,1,1),\ (1,3,1,0)\}.$$

(4 marks)

(e) For L and L' in (d), determine whether or not there exists $S \in Sp(4)$ such that $S(\mathbb{R}^2 \times 0) = \mathbb{R}^2 \times 0$, $S(0 \times \mathbb{R}^2) = 0 \times \mathbb{R}^2$, and $S(L) = L'$. **(6 marks)**

End of Question Paper

List of Formulae for Section A

For a curve on $\mathbb{R}^2$ parametrised by $\mathbf{x}(t) = (x(t), y(t))$:

- arc length from $\mathbf{x}(a)$ to $\mathbf{x}(b)$

$$\int_a^b ||\mathbf{x}'(t)||\, dt$$

- curvature

$$k(t) = \frac{x'(t)y''(t) - y'(t)x''(t)}{[x'(t)^2 + y'(t)^2]^{3/2}}$$

For a surface in $\mathbb{R}^3$ parametrised by $\mathbf{x}(u, v)$:

- first fundamental form

$$Edu^2 + 2Fdudv + Gdv^2, \quad E = \mathbf{x}_u \cdot \mathbf{x}_u,\ F = \mathbf{x}_u \cdot \mathbf{x}_v,\ G = \mathbf{x}_v \cdot \mathbf{x}_v$$

- surface areas

$$\iint \sqrt{EG - F^2}\, dudv$$

- second fundamental form

$$Ldu^2 + 2Mdudv + Ndv^2, \quad L = \mathbf{x}_{uu} \cdot \mathbf{n},\ M = \mathbf{x}_{uv} \cdot \mathbf{n},\ N = \mathbf{x}_{vv} \cdot \mathbf{n}$$

$$\text{where } \mathbf{n} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{||\mathbf{x}_u \times \mathbf{x}_v||}$$

- Weingarten matrix

$$W = \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \begin{bmatrix} L & M \\ M & N \end{bmatrix}$$

- mean and Gaussian curvatures

$$H = \frac{1}{2}\operatorname{tr} W, \qquad K = \det W$$

The Gauss-Bonnet formula for a compact region R on a surface:

$$\iint_R K dA + \int_{\partial R} k_g ds + \sum \text{turning angles} = 2\pi\chi(R)$$