The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS Spring Semester 2012-2013

Analytical Dynamics and Classical Field Theory 3 Hours

Answer five questions. If you answer more than five questions, only your best five will be counted.

1 (i) Write down the Euler-Lagrange equation and define all quantities which appear in this equation. ***(5 marks)***

(ii) The Lagrange–function of a system with two degrees of freedom is given by

$$L(x(t), y(t), \dot{x}(t), \dot{y}(t)) = \dot{x}^2 - \dot{y}^2 + \mu(x+y)^2,$$

where $\dot{x} = dx/dt$ and $\dot{y} = dy/dt$. Find $x(t)$ and $y(t)$.

Hint: To solve the equations of motion, it is easier to work with $u(t)$ and $v(t)$ defined by $x = u + v$ and $y = u - v$ than to work with x and y. ***(15 marks)***

2 (i) The Lagrange function of a system is given by $L(q_i, \dot{q}_i, t)$, where $i = 1, 2, 3, \ldots N$ and N is the number of degrees of freedom, q_i are the generalised coordinates and $\dot{q}_i$ are the corresponding generalised velocities. Show that

$$\frac{d}{dt}\left(L - \sum_{i=1}^{N} \dot{q}_i \frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial t} = 0$$

(8 marks)

(ii) Two ideal springs with spring constant k are attached to a point mass m. The other end points of the springs are each attached to a wall. The mass can only move in the horizontal x direction.

a) What is the Lagrange function for this system? Assume that the springs have their natural extensions if $x = 0$. You are given that the potential energy of one spring is $V(x) = \frac{1}{2}kx^2$. ***(2 marks)***

b) Write down the Euler-Lagrange equation and find the angular frequency of the oscillations around $x = 0$. ***(5 marks)***

c) Write down the Hamilton function for the system. Using the Hamilton equations, show that the result is consistent with what you found in part (ii) b). ***(5 marks)***

3 Consider a mass m on the end of a spring of natural length L and a spring constant k. Let y be the vertical coordinate of the mass as measured from the top of the spring. Assume that the mass can only move up and down in the vertical direction. You are given that the potential energy of a spring can be written as $V_s = \frac{1}{2}k\,(y - L)^2$. The potential energy in a gravitational field can be written as $V_g = -mgy$.

(i) Write down the Lagrange–function and find the canonical momentum. Find the Hamilton–function, **without** using the fact that $H = T + V$, where T is the kinetic energy and V is the potential energy. ***(4 marks)***

(ii) Using Hamilton equations, find the general solution for $y(t)$. Show that the frequency of the oscillations is independent of whether the gravitational field is present or not. ***(14 marks)***

(iii) What is the natural extension of the spring in the gravitational field? ***(2 marks)***

4 (i) Define the Poisson bracket $\{f, g\}$ between two functions $f(q_i, p_i)$ and $g(q_i, p_i)$, where $i = 1, 2, 3...N$, q_i are generalised coordinates and p_i are the corresponding momenta. ***(2 marks)***

(ii) Show explicitly that the Poisson brackets obey the following (f, g and h are functions of the generalised coordinates and momenta): ***(4 marks)***

(a) $\{f, g\} = -\{g, f\}$

(b) $\{f, g+h\} = \{f, g\} + \{f, h\}$

(iii) A particle of mass m moves under a potential, which depends only on the distance from the origin, i.e. $V(x, y, z) = V(r)$, where $r = \sqrt{x^2 + y^2 + z^2}$. The angular momentum is the vector defined by $\mathbf{L} = \mathbf{r} \times \mathbf{p}$. Show that $\{L_z, H\} = 0$. Explain why $\{L_x, H\} = 0$ and $\{L_y, H\} = 0$. Interpret your results. ***(14 marks)***

5 The electromagnetic field strength tensor is defined to be $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

(i) How does this tensor transform under general Lorentz transformations? ***(3 marks)***

(ii) $F_{\mu\nu}$ can be written in terms of the electric field $\mathbf{E} = (E_x, E_y, E_z)$ and magnetic field $\mathbf{B} = (B_x, B_y, B_z)$ as follows

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}.$$

Assume that in an inertial frame, called frame A, $F_{\mu\nu}$ has the form above. Consider now a second inertial frame, called B, moving with velocity $\mathbf{v} = (v, 0, 0)$ in the x–direction relative to frame A. The Lorentz–transformation between frames A and B is given by the matrix

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$. Assume that in frame B the electromagnetic field strength tensor has the same form as in frame A, i.e. it can be written as

$$F'_{\mu\nu} = \begin{pmatrix} 0 & -E'_x & -E'_y & -E'_z \\ E'_x & 0 & B'_z & -B'_y \\ E'_y & -B'_z & 0 & B'_x \\ E'_z & B'_y & -B'_x & 0 \end{pmatrix}.$$

Show that $E'_x = E_x$, $B'_x = B_x$ and $B'_y = \gamma(B_y + \beta E_z)$. ***(17 marks)***

6 The Lagrangian $\mathcal{L}$ for two scalar fields ϕ_1 and ϕ_2 with mass m is given by

$$\mathcal{L} = \frac{1}{2}\left(\eta^{\mu\nu}(\partial_\mu\phi_1)(\partial_\nu\phi_1) + \eta^{\mu\nu}(\partial_\mu\phi_2)(\partial_\nu\phi_2) - m^2\left(\phi_1^2 + \phi_2^2\right)\right).$$

(i) Find the equations of motion for ϕ_1 and ϕ_2. ***(3 marks)***

(ii) Consider the following transformation with α constant:

$$\phi_1 \to \phi_1' = \phi_1 \cos\alpha - \phi_2 \sin\alpha$$
$$\phi_2 \to \phi_2' = \phi_1 \sin\alpha + \phi_2 \cos\alpha$$

Show that the Lagrangian $\mathcal{L}$ above is invariant under this transformation, i.e. $\mathcal{L}' = \mathcal{L}$. ***(9 marks)***

(iii) Find the infinitesimal transformation (i.e. for $\alpha \to 0$). ***(2 marks)***

Hint: keep only the leading order terms when considering the limit $\alpha \to 0$.

(iv) Find the conserved Noether current. You are given that, in general, the Noether current is given by

$$j^\mu = \sum_{i=1}^{N} \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\delta\phi_i - \mathcal{J}^\mu,$$

where under the infinitesimal transformation $\phi_i(x) \to \phi_i'(x) = \phi_i(x) + \beta\delta\phi_i(x)$ the Lagrangian transforms as $\mathcal{L} \to \mathcal{L}' = \mathcal{L} + \beta\partial_\mu\mathcal{J}^\mu$, where β is a small parameter.

(6 marks)

End of Question Paper