Data provided: Formulae sheet

CIV340

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS

Autumn Semester 2013-2014

Computational Engineering Mathematics

Three hours

Marks will be awarded for your best FOUR answers

1 (i) The second order pde

$$A\frac{\partial^2\Phi}{\partial x^2} + B\frac{\partial^2\Phi}{\partial x\partial y} + C\frac{\partial^2\Phi}{\partial y^2} + D\frac{\partial\Phi}{\partial x} + E\frac{\partial\Phi}{\partial y} + F = 0,$$

where A, B, C, D, E and F are arbitrary constants, can be classified as being either elliptic, parabolic or hyperbolic according to the values of A, B and C.

(a) For each of the three types of pde, give a simple example of a physical system which is modelled by that type. ***(3 marks)***

(b) State what conditions on A, B and C are required for the equation above to be elliptic, and state what additional conditions are then required to solve the problem. ***(3 marks)***

(ii) The one-dimensional diffusion equation, together with necessary additional conditions, is given by

$$\frac{\partial U}{\partial t} = \alpha\frac{\partial^2 U}{\partial x^2}, \;\; U(x,0) = f(x), \;\; U(0,t) = a, \;\; U(1,t) = b$$

where α is the *diffusion coefficient.* Using the standard notation that $U_{ij} \equiv U(x_i, t_j)$ together with the conventions that $i = 1$ and $i = n$ correspond to $x = 0$ and $x = 1$ respectively and that $j = 1$ corresponds to $t = 0$, use the standard finite difference approximations, given on the formulae sheet, together with the notation $k = \Delta t/\Delta x^2$, to derive the *explicit scheme*

$$U_{ij} = \alpha k(U_{i+1j-1} + U_{i-1j-1}) + (1 - 2\alpha k)U_{ij-1} \;\;, i = 2, .., n-1, \;\; j = 2, 3, ..$$

which approximates the differential equation. ***(3 marks)***

(iii) The diffusion equation is to be solved (approximately) over the range $0 \leq x \leq 1$ for the temperature distribution along a given steel billet with boundary conditions $U(0,t) = 20^oC$ and $U(1,t) = 100^oC$ and initial conditions $U(x,0) = 80x^2 + 20$, where it is assumed that the units have been normalized so that $\alpha = 1$. Assuming that we use $\Delta x = 0.025$ and $\Delta t = 0.0005$, then write a program which uses the explicit scheme to generate the approximate solution up to $t = 1$. You may use a programming language from amongst ***Scilab, Matlab, Fortran, Python or IDL***. State clearly which language you are using. ***(11 marks)***

2 (i) A vanishingly small force, $\Delta\mathbf{f}$, acts on a surface of vanishingly small area, ΔA, drawn on the interior of a solid body. Using a diagram to clarify things, define what is meant by the *stress* at a point P in ΔA and explain, briefly, why a complete mathematical description of *stress* requires it to be defined as a two-index tensor. ***(6 marks)***

(ii) A concrete slab, of unit thickness in the z-direction, is loaded with body-forces $\mathbf{f}$ and is in a state of plane stress so that $\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = \sigma_{zx} = \sigma_{zy} = F_z = 0$. By considering only the balance of forces in the x-direction, use a diagram to derive the x-component of the equations of static equilibrium and hence infer for the general case (i.e. when σ_{zj}, σ_{iz}, $F_z \neq 0$) the full set of force-balance equations for a three-dimensional body.

(14 marks)

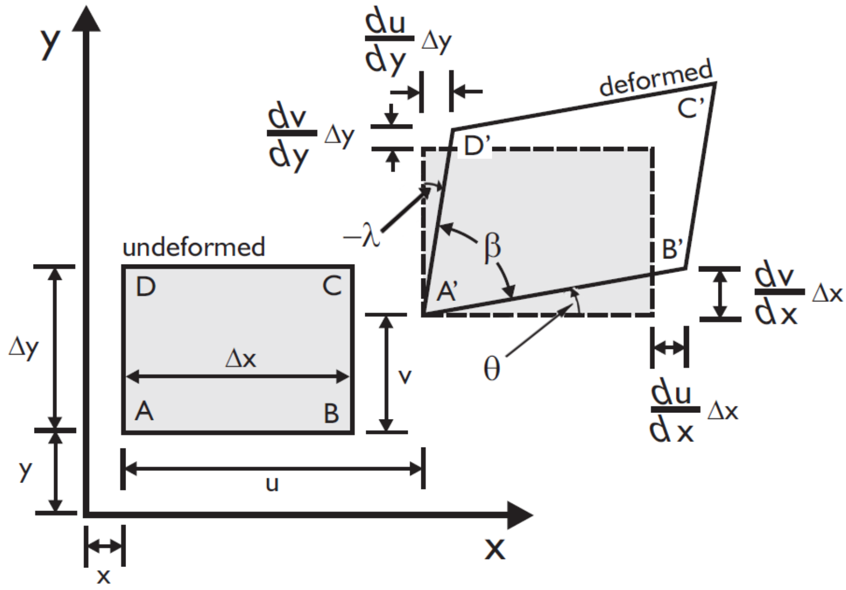

Figure 1: Two-dimensional strain

3 (i) By reference to Figure 1, define the normal strain, ε_{yy}, and the engineering shear strain, γ_{yx}, and hence show that

$$\varepsilon_{yy} = \frac{\partial v}{\partial y}, \quad \gamma_{yx} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y},$$

where u is the displacement in the x-direction and v is the displacement in the y-direction of the body ABCD due to the stress forces acting on its surfaces. ***(12 marks)***

3 (continued)

(ii) The elastic constitutive matrix applying to the *engineering* strains for an isotropic material is given by

$$C = \begin{bmatrix} (\lambda + 2\mu) & \lambda & \lambda & 0 & 0 & 0 \\ & (\lambda + 2\mu) & \lambda & 0 & 0 & 0 \\ & & (\lambda + 2\mu) & 0 & 0 & 0 \\ & & & \mu & 0 & 0 \\ & \text{sym} & & & \mu & 0 \\ & & & & & \mu \end{bmatrix}$$

where

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)}.$$

Given further that $E = 33.0GPa$, $\nu = 0.185$, and that a state of strain defined by $\varepsilon_{xx} = 1010 \times 10^{-6}$, $\varepsilon_{yy} = -0.28\varepsilon_{xx}$, $\varepsilon_{zz} = -0.19\varepsilon_{xx}$, $\varepsilon_{xy} = 227 \times 10^{-6}$, $\varepsilon_{yz} = 427 \times 10^{-6}$ and $\varepsilon_{zx} = -71 \times 10^{-6}$ exists at a point in a given isotropic material, calculate the corresponding state of stress at the point. ***(8 marks)***

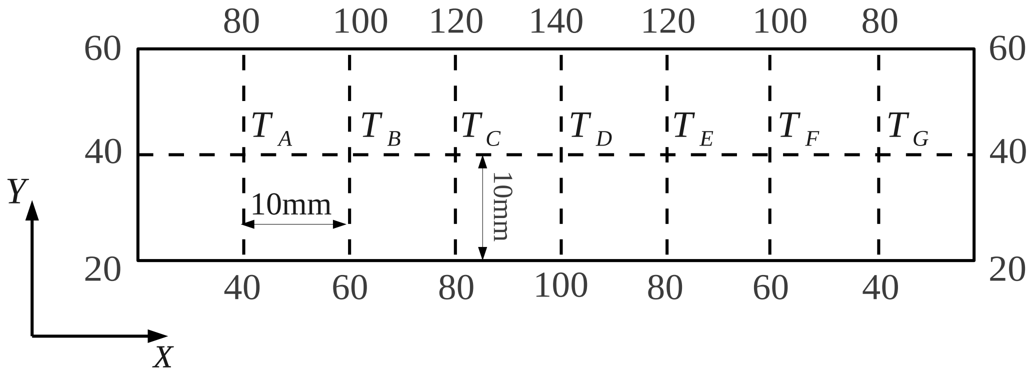

Figure 2: A rectangular plate with temperature defined on the boundaries.

4 Figure 2 shows a rectangular plate made of a homogeneous isotropic material. The temperature distribution in this plate satisfies the indicated boundary conditions (given in degrees centigrade) and has reached a steady-state condition so that it is described by Laplace's equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0.$$

(i) Draw a sketch of the solution domain showing clearly the line of symmetry for the temperature distribution and indicating which of the unknown temperatures are equal to each other. ***(4 marks)***

(ii) Use the finite difference formulae on the formulae sheet to formulate the finite difference equations required to find estimates of the nodal temperatures, T_A, T_B, T_C and T_D. ***(10 marks)***

(iii) Express these finite difference equations in the form $A\mathbf{T} = \mathbf{B}$ where A is a 4×4 matrix, $\mathbf{T} = (T_A, T_B, T_C, T_D)^T$ and $\mathbf{B} = (-160, -160, -200, -240)^T$ is a 4×1 column vector. Find matrix A, hence, given that

$$A^{-1} \approx \begin{bmatrix} -0.27 & -0.07 & -0.02 & -0.01 \\ -0.07 & -0.29 & -0.08 & -0.02 \\ -0.02 & -0.08 & -0.31 & -0.08 \\ -0.01 & -0.04 & -0.15 & -0.29 \end{bmatrix}$$

estimate T_A, T_B, T_C and T_D correct to one degree. ***(6 marks)***

5 The velocity field in an unsteady moving fluid is given by $\mathbf{V} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$, where $u \equiv u(x, y, z, t)$, $v \equiv v(x, y, z, t)$ and $w \equiv (x, y, z, t)$.

(i) By considering the density, $\rho(x, y, z, t)$, and an infinitesimal control volume, $\delta\mathcal{V}$, in the fluid moving from a point (x_1, y_1, z_1) at time t_1 to a point (x_2, y_2, z_2) at time t_2, then derive the substantial (or total) derivative

$$\frac{D\rho}{Dt} = \frac{\partial\rho}{\partial t} + u\,\frac{\partial\rho}{\partial x} + v\,\frac{\partial\rho}{\partial y} + w\,\frac{\partial\rho}{\partial z}$$

of ρ and interpret the meanings of the first two terms on the right-hand side of this latter expression. ***(12 marks)***

(ii) $\mathbf{G}$, the curl of a vector field $\mathbf{F}$, may be expressed in index notation as

$$G_i = \varepsilon_{ijk}\frac{\partial F_k}{\partial x_j} \text{ where } i, j, k \text{ may each take any of the values } 1, 2, 3.$$

The components of the Levi-Civita tensor $\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1$, $\varepsilon_{132} = \varepsilon_{213} = \varepsilon_{321} = -1$, and are zero otherwise. A useful relation between the Levi-Civita tensor and the Kronecker delta is

$$\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$$

The vorticity of a fluid is $\vec{\omega} = \nabla \times \mathbf{u}$, the curl of the velocity field $\mathbf{u}$. The curl of the vorticity is $\nabla \times \vec{\omega}$. Verify by index notation or otherwise that the curl of the vorticity may be expressed as $\nabla(\nabla \cdot \mathbf{u}) - \nabla^2\mathbf{u}$. ***(8 marks)***

End of Question Paper

Formulae Sheet

Notation:

$$U(x_i,\ t_j) \equiv U_{i\,j}$$

Forward difference formula for $\partial U/\partial t$:

$$\frac{\partial U}{\partial t} \approx \frac{U_{i,j+1} - U_{ij}}{\Delta t}$$

Backward difference formula for $\partial U/\partial t$:

$$\frac{\partial U}{\partial t} \approx \frac{U_{ij} - U_{i,j-1}}{\Delta t}$$

Central difference formula for $\partial U/\partial x$:

$$\frac{\partial U}{\partial x} \approx \frac{U_{i+1,j} - U_{i-1,j}}{2\Delta x}$$

Central difference formula for $\partial^2 U/\partial x^2$:

$$\frac{\partial^2 U}{\partial x^2} \approx \frac{U_{i+1,j} - 2U_{ij} + U_{i-1,j}}{\Delta x^2}$$