The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2013–14**

Mathematics Core I **2 hours**

Attempt all the questions. The allocation of marks is shown in brackets.

Total marks 60.

1 Let $A = \{2, 4, 6\}$ and $B = \{3, 4, 5, 6, 7, 8\}$.

(i) What are the cardinalities (sizes) of the sets $A \cap B$ and $A \cup B$? ***(2 marks)***

(ii) If $f : A \to B$ is a function, with image $f(A)$, what are the possibilities for $|f(A)|$, the cardinality of the image? For each possible value of $|f(A)|$, give an example of a function $f : A \to B$ for which $|f(A)|$ has that value, in each case defining f by specifying $f(2), f(4)$ and $f(6)$. ***(2 marks)***

(iii) How many different functions $f : A \to B$ are there? How many possibilities are there for the image $f(A)$, when all possible choices for f are considered? ***(3 marks)***

2 For fixed $r \neq 1$, and any $n \in \mathbb{N}$, we define $S_n := \sum_{m=0}^{n-1} r^m$. Prove, from first principles, that if $k \in \mathbb{N}$ and if $S_k = \dfrac{1 - r^k}{1 - r}$, then $S_{k+1} = \dfrac{1 - r^{k+1}}{1 - r}$. How do you complete this to a proof that $S_n = \dfrac{1 - r^n}{1 - r}$ for all $n \in \mathbb{N}$? ***(4 marks)***

3 If $y = y(u(x))$, what formula does the Chain Rule give for $\dfrac{dy}{dx}$? Now let $y = e^{4x}$. Letting $u = 4x$, find $\dfrac{dy}{dx}$, showing carefully the steps in your reasoning. Choosing instead $u = e^x$, find $\dfrac{dy}{dx}$ another way, and check that you get the same answer. ***(4 marks)***

4 Let $y = (8x^3 - 12x)e^{-x^2/2}$. Show that y is a solution to the differential equation

$$y'' + (7 - x^2)y = 0.$$

(5 marks)

5 (i) State, without proof, formulas for the derivatives of $\sin x$ and $\cos x$. ***(1 mark)***

(ii) Let $f(x) := \cos^2(x) + \sin^2(x)$. Without assuming any trigonometric identity, but using only the formulas you quoted in (i), prove that $f(x)$ is constant, and determine the value of this constant. ***(2 marks)***

(iii) Suppose that $y = u/v$ (with $v \neq 0$), where u and v are functions of x. Starting from $y + \Delta y = (u + \Delta u)/(v + \Delta v)$, obtain the Quotient Rule formula for $\dfrac{dy}{dx}$. ***(2 marks)***

(iv) Using the formulas you quoted in (i), deduce a formula for the derivative of $\tan x$, expressing your answer in terms of $\tan x$. ***(2 marks)***

(v) Let $y = \tan^{-1}(x)$. What is the range, or image, of this function? Sketch its graph, marking in the points for which $x = \pm 1$. Using what you have already proved, obtain a formula for $\dfrac{dy}{dx}$. For what value of x is this derivative a maximum? ***(5 marks)***

6 (i) Calculate the integral $\displaystyle\int_{-1}^{1} x \tan^{-1}(x)\, dx$. ***(4 marks)***

(ii) Calculate the integral $\displaystyle\int_{0}^{2} \frac{1}{4 + 2x + x^2}\, dx$. ***(4 marks)***

7 Write down the Maclaurin series for e^x and for $\cosh x$. What is $\cosh(ix)$? ***(3 marks)***

8 (i) Calculate the complex numbers $z + \overline{w}$, zw and $z/\overline{w}$, where $z = 2 + i$ and $w = 3 - 2i$. ***(3 marks)***

(ii) On the Argand diagram, mark a point representing any complex number α you choose, subject only to the requirement that $\alpha \notin \mathbb{R}$. Do not say what it is; just label it "α". Now, on the same diagram, mark clearly the complex numbers 2α, $i\alpha$ and $(2+i)\alpha$. ***(3 marks)***

(iii) Express the complex number $z = -\sqrt{3} + i$ in the form $z = re^{i\theta}$, with $r \geq 0$ and $-\pi < \theta \leq \pi$. What is the smallest integer $n > 0$ such that $z^n \in \mathbb{R}$? ***(3 marks)***

9 (i) Find the general solution to the differential equation $y'' + 4y' + 4y = 0$. ***(2 marks)***

(ii) Find the general solution to the differential equation $y'' + 4y' + 4y = \sin t$. ***(3 marks)***

(iii) Find the solution to $y'' + 4y' + 4y = e^{-2t}$ such that $y(0) = y'(0) = 0$. ***(3 marks)***

End of Question Paper