The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2013–14**

Advanced Calculus - MAS202 **2 hours**

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) Let f be the periodic function with period 2π such that $f(x) = x - x^2$ for $-\pi < x < \pi$.

(a) Sketch the graph of f. Is the function f odd, even or neither? Justify your answer. ***(4 marks)***

(b) Calculate all the coefficients in the Fourier series for f. ***(10 marks)***

(c) What can you deduce by plugging in $x = 0$? ***(5 marks)***

(ii) A function F is defined by the formula

$$F(x) = \int_{\cos(x)}^{x^2+1} e^{t^2(x+1)}\, dt.$$

Write down an expression for the derivative $\frac{dF}{dx}$. (Do not attempt to evaluate the integral in your expression.) ***(6 marks)***

2 (i) Let $\omega = P(x,y)dx + Q(x,y)dy$ be a differential form.

(a) Explain what you mean by an *exact* differential form. ***(3 marks)***

(b) State the exactness criterion for a differential form, being careful to include any conditions needed for its validity. ***(4 marks)***

(c) Prove the exactness criterion. ***(10 marks)***

(ii) Show that the differential $\omega = (y + \cos(x))dx + xdy$ is exact and find a potential function f for ω. ***(8 marks)***

3 In this question you are asked to find a solution $u = u(x,t)$ to the heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

for a thin rod of length π represented as the interval $0 \le x \le \pi$ on the x-axis that satisfies the conditions

$$(*) \qquad u(0,t) = u(\pi,t) = 0 \ \text{ for all } t \ge 0.$$

(i) As the first step, verify that for each natural number $n \ge 1$ the function

$$u(x,t) = \sin(nx)e^{-c^2n^2t}$$

is a solution of the heat equation that satisfies the condition (*).

(7 marks)

(ii) Using (i), find a solution to the heat equation which has the initial temperature distribution $u(x,0) = f(x)$ where $f(x) = e^x$ for $0 < x < \pi$.

(15 marks)

(iii) Find the rate of cooling down of the rod at the point $x = 1$ and at the moment $t = 1$, i.e., $\frac{\partial u}{\partial t}|_{x=1,t=1}$.

(3 marks)

4 (i) The random variables X and Y have joint density function

$$f(x,y) = \begin{cases} \frac{2}{\pi}(x^2+y^2) & \text{if } x^2+y^2 \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dxdy \;=\; 1.$$

(b) Calculate the probability $P\,(X \ge 0)$.

(c) Calculate the probability $P\,(Y - X \le 0)$.

(12 marks)

(ii) State Green's Theorem, being careful to include any conditions needed for its validity. Hence evaluate

$$\int_C (x^2+y^3+4+xe^{x^2+y^2})dx + (x^2+y^3+4+ye^{x^2+y^2})dy,$$

where C is the triangular path with vertices $(0,0)$, $(1,1)$ and $(1,0)$, described in the anticlockwise direction. ***(8 marks)***

(iii) Let D be a region of the (x,y)-plane whose boundary is a positively oriented closed path C. Prove that the area of D is equal to

$$\int_C (2012y+xe^{x^2+y^2})dx + (2013x+ye^{x^2+y^2})dy.$$

(5 marks)

End of Question Paper