The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2013–14**

MECHANICS **2 hours**

Marks will be awarded for your best THREE answers.

1 (i) Show that the transformation from Cartesian to plane polar coordinates, as defined by $x = r\cos\phi$, $y = r\sin\phi$, has the Jacobian $J = r$, where $J \equiv \partial(x, y)/\partial(r, \phi)$. ***(4 marks)***

(ii) Consider a thin disk of radius R, mass M and uniform density. Use integration to show that:

(a) The area of the disk is πR^2. ***(3 marks)***

(b) The moment of inertia of the disk with respect to a perpendicular axis passing through the centre of the disc is $\frac{1}{2}MR^2$. ***(5 marks)***

(iii) Now consider a uniform sphere of radius a and mass m. By considering the sphere to be made up of very thin disks, or otherwise, find the moment of inertia of the sphere with respect to its axis of symmetry. ***(10 marks)***

(iv) Use the parallel axes theorem to find the moment of inertia with respect to an axis which is tangent to a point on the surface of the sphere. ***(3 marks)***

2 A particle has mass m, position vector $\mathbf{r}(t)$, kinetic energy T and specific angular momentum $\mathbf{h}$, given by

$$T = \frac{1}{2} m \, \dot{\mathbf{r}} \cdot \dot{\mathbf{r}}, \qquad \mathbf{h} = \mathbf{r} \times \dot{\mathbf{r}},$$

where $\dot{\mathbf{r}} \equiv d\mathbf{r}/dt$. It moves under a position-dependent force $\mathbf{F}(\mathbf{r})$.

(i) Use Newton's 2nd law to show that the rate of change of kinetic energy due to the force is $\dot{T} = \mathbf{F} \cdot \dot{\mathbf{r}}$. ***(4 marks)***

(ii) (a) Show that the change in kinetic energy along a path P is equal to the line integral

$$W = \int_P \mathbf{F} \cdot d\mathbf{r}$$

where W is the work done on the particle by the force. ***(5 marks)***

(b) If there exists a scalar function $V(\mathbf{r})$ such that $\mathbf{F} = -\nabla V$, then $\mathbf{F}$ is *conservative.* What can be deduced about the line integral of a conservative force? ***(2 marks)***

(c) Show that any central force is conservative. (N.B. a central force is of the form $\mathbf{F} = F(r)\hat{\mathbf{r}}$, where $\hat{\mathbf{r}} \equiv r^{-1}\mathbf{r}$). ***(4 marks)***

(iii) (a) Show that angular momentum is conserved under a central force, by showing that $\dot{\mathbf{h}} = \mathbf{0}$. ***(5 marks)***

(b) State Kepler's second law of planetary motion. Explain why, in principle, the second law would apply for *any* central force. ***(5 marks)***

3 Consider a cylindrically-symmetric uniform body of mass m defined by the volume enclosed by the three surfaces $z = 0$, $z = a$ and $x^2 + y^2 = a^2 \left(1 + \frac{1}{2}\cos(\pi z/a)\right)$.

(i) Draw a labelled sketch of the body in the Cartesian coordinate system. ***(4 marks)***

(ii) (a) Show that its volume is πa^3. ***(4 marks)***

(b) Show that its moment of inertia about its axis of symmetry is $I = \frac{9}{16}ma^2$. ***(5 marks)***

(c) Find the height of the centre of mass above the base. ***(6 marks)***

(iii) Now suppose this body rotating around the axis of symmetry at an angular speed ω.

(a) Write down an expression for the body's rotational kinetic energy in terms of I and ω. ***(2 marks)***

(b) If the body has the same rotational kinetic energy as a solid cylinder of identical mass whose height and base are a, which is rotating faster? Find the ratio of angular speeds. ***(4 marks)***

4 A particle of mass m described by plane polar coordinates $(r(t), \phi(t))$ moves in a Newtonian gravitational potential. The conservation of energy equation is

$$\frac{E}{m} = -\frac{GM}{r} + \frac{1}{2}\left(\dot{r}^2 + \frac{h^2}{r^2}\right) \tag{1}$$

where E and h are constants of motion, and G is Newton's gravitational constant. Assume that the masses m and M are constant.

(i) By considering the radial derivative of Eq. (1), and by using the chain rule $\frac{d}{dr} = \frac{1}{\dot{r}}\frac{d}{dt}$ as appropriate, show that the radial acceleration is

$$\ddot{r} = -\frac{GM}{r^2} + \frac{h^2}{r^3}$$

Interpret the two terms on the right-hand side with reference to Newton's 2nd law in one dimension. ***(5 marks)***

(ii) Now consider circular orbits, for which $\dot{r} = 0 = \ddot{r}$.

(a) Find an expression for h in terms of r. ***(3 marks)***

(b) Find the angular frequency $\dot{\phi} = h/r^2$, and hence show that the orbital period is $2\pi\sqrt{r^3/(GM)}$ ***(4 marks)***

(c) The Earth-Sun distance is approximately 1.5×10^{11} metres, and $G \approx 6.67 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$. Use the orbital period equation to estimate the mass of the Sun in kilograms. ***(4 marks)***

(d) Show that the total energy is

$$E = -\frac{1}{2}\frac{GMm}{r},$$

that is, half the potential energy. ***(3 marks)***

(iii) A more general solution describing non-circular orbits is

$$r(\phi) = \frac{r_0}{1 + e\cos(\phi)}$$

where e and r_0 are constants of motion. Sketch several solutions in the (r, ϕ) plane for $r_0 = 1$ and a range of $e \in [0, 1)$. State Kepler's first law of planetary motion, and give an interpretation of e. ***(6 marks)***

End of Question Paper