The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2013–14**

Topics in Number Theory (Level 2) **2 hours**

Attempt all the questions. The allocation of marks is shown in brackets.

No credit will be given for solutions which rely solely on the use of a calculator. Your solutions should give enough details to make it clear how you arrived at your answers.

1 (i) You publish $(n, e) = (221, 65)$ in the RSA directory and receive 2. Decode it. ***(10 marks)***

(ii) What is the remainder when

$$2013^{2014\times2015\times2017}$$

is divided by 13? ***(7 marks)***

(iii) Find the solutions to the system of congruences

$$\begin{aligned} 2x &\equiv 5 \mod 7, \\ 4x &\equiv 8 \mod 11. \end{aligned}$$

(8 marks)

2 (i) State the *Law of Quadratic Reciprocity.* ***(2 marks)***

(ii) Solve the congruence

$$x^2 + 9x + 18 \equiv 0 \mod 95.$$ ***(10 marks)***

(iii) Find $\left(\frac{98!}{101}\right)$. ***(8 marks)***

(iv) Show that

$$36 \times 27! + 25$$

is divisible by 31. (No credit will be given for a solution that does not use Wilson's Theorem) ***(5 marks)***

3 (i) Expand $\sqrt{17}$ as a continued fraction, find a convergent of $\sqrt{17}$ which differs from it by less than 10^{-6}, and find two solutions of the Pell equation

$$x^2 - 17y^2 = 1.$$

(10 marks)

(ii) Express the continued fraction $[1; 2, \overline{3}]$ in the form $a + b\sqrt{c}$ where a, b are rational numbers and c is a positive integer. ***(5 marks)***

(iii) State Gauss' Lemma and *using it* find $\left(\frac{4}{13}\right)$ (no credit will be given without using Gauss' Lemma). ***(6 marks)***

(iv) Let $p > 2$ be a prime number. Prove that

$$\left(\frac{2}{p}\right) = (-1)^{\frac{p-1}{2} - [\frac{p}{4}]}.$$

(4 marks)

4 (i) Give a definition of a perfect number. Which of the numbers $6, 7$ and 8 are perfect. Justify your response. ***(3 marks)***

(ii) For a positive integer n, let $\sigma(n) = \sum_{d|n} d$. Show that

$$\sigma(mn) = \sigma(n)\sigma(n)$$

provided $(m, n) = 1$. ***(5 marks)***

(iii) State the criterion that describes *even perfect* numbers. Prove the criterion. ***(12 marks)***

(iv) Give a definition of the Fibonacci sequence $(f_i)_{i \geq 1}$. Let

$$\mathcal{A} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Show that, for all natural numbers $m \geq 1$ and $n \geq 2$,

$$\begin{pmatrix} f_{n+m} \\ f_{n+m-1} \end{pmatrix} = \mathcal{A}^m \begin{pmatrix} f_n \\ f_{n-1} \end{pmatrix}.$$

(5 marks)

End of Question Paper