Data provided: formula sheet

MAS241

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2013–14**

Mathematics II (Electrical) **2 hours**

Attempt all the questions. The allocation of marks is shown in brackets.

Please leave this exam paper on your desk
Do not remove it from the hall

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to be completed by student

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1 (i) The electrical charge in a circuit is described by

$$q''(t) + 4q'(t) + 8q(t) = \delta(t)$$

subject to the initial conditions $q(0) = q'(0) = 0$.

(a) Show that the Laplace transform of $q(t)$ is given by

$$Q(s) = \frac{1}{2}\left(\frac{2}{(s+2)^2 + 2^2}\right).$$

(7 marks)

(b) Use the inverse Laplace transform to determine the charge $q(t)$ at time $t > 0$. **(5 marks)**

(ii) Let $f(t) = e^{-|t|}$ and $g(t) = \delta(t-1)$.

(a) Use the convolution and time shift property of the Fourier transform to show that

$$\mathcal{F}\{f * g(t)\} = \frac{2e^{-j\omega}}{1+\omega^2}.$$

(4 marks)

(b) Use the inverse Fourier transform to calculate $f * g(t)$. Verify your result by the direct calculation of $f * g(t)$. **(4 marks)**

2 (i) Let $f : [0, \pi] \to \mathbb{R}$ be defined by $f(t) = \pi - t$. Find the Fourier sine series of $f(t)$. **(15 marks)**

(ii) Sketch the graph of the Fourier sine series of $f(t)$ over the interval $[-\pi, \pi]$. Indicate where the Fourier sine series agrees with the odd extension of $f(t)$ in the interval $[-\pi, \pi]$. **(5 marks)**

3 (i) Let $f(x, y) = x^2 + y^2 + \sin\left(x^2\right)$. Find $f_x(x, y)$ and $f_y(x, y)$. **(3 marks)**

(ii) Find and classify *all* the critical points of the function

$$f(x, y) = 2x^5 - y^5 - 10x + 5y$$

(10 marks)

(iii) Find $\dfrac{\partial f}{\partial v}$ when $f(x, y) = y(x + 2y)$, $x(u, v) = \cos uv$ and $y(u, v) = e^{u+v}$.

(7 marks)

4 (i) Let $D \subset \mathbb{R}^2$ be the region bounded by the x-axis, y-axis, and the graph of the function $y = 1 - x^2$. Sketch the region D and calculate its area.

(7 marks)

(ii) Let R be a cylinder bounded by the xy-plane, the plane $z = h$, and by the surface $x^2 + y^2 = R^2$. The mass density of the cylinder material is $\rho = \rho_0 e^{-(r/R)^2 + z/h}$. Calculate the cylinder mass M. **(13 marks)**

Hint: Use cylindrical coordinates r, θ, z.

5 (i) Let $f(x, y) = x^2 - x^2 y - 2y^2$.

(a) Calculate the directional derivative of $f(x, y)$ at $(3, -1)$ in the direction of $\mathbf{v} = (4, 3)$. **(5 marks)**

(b) In what direction is the graph of $f(x, y)$ most rapidly increasing at the point $(3, -1)$, and what is the maximum rate of increase?

(5 marks)

(ii) Let $f(x, y, z) = xy$ and $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field defined by

$$\mathbf{F} = (xy, xz, yz).$$

Verify by the direct calculation that

$$\nabla \times (f\,\mathbf{F}) = f\,\nabla \times \mathbf{F} + (\nabla f) \times \mathbf{F}.$$

(10 marks)

End of Question Paper

MAS241 FORMULA SHEET

Laplace transform:

The Laplace transform of a function $f(t)$ is given by:

$$\mathcal{L}\{f(t)\}(s) := \int_0^\infty e^{-st} f(t) dt.$$

Properties of the Laplace transform: $\mathcal{L}\{f(t)\} = F(s)$ in the following table.

$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$	linearity
$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$	differentiation w.r.t. t
$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$	second differentiation w.r.t. t
$\mathcal{L}\{e^{-kt}f(t)\} = F(k+s)$	frequency shift
$\mathcal{L}\{f(t-a)H(t-a)\} = e^{-as}F(s)$ (for $a > 0$)	time shift
$\mathcal{L}\{f(at)\} = \frac{1}{a}F(\frac{s}{a})$ (for $a > 0$)	scaling
$\mathcal{L}\{f * g(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$ (for $f(t), g(t)$ causal)	convolution

Table of standard Laplace transforms:

$f(t)$	$\mathcal{L}\{f(t)\}(s)$	Region of validity
t^n (for $n \geq 0$)	$\frac{n!}{s^{n+1}}$	$Re(s) > 0$
$\sin(kt)$	$\frac{k}{s^2+k^2}$	$Re(s) > 0$
$\cos(kt)$	$\frac{s}{s^2+k^2}$	$Re(s) > 0$
$H(t-T)$ (for $T \geq 0$)	$\frac{e^{-sT}}{s}$	$Re(s) > 0$
$\delta(t-T)$ (for $T \geq 0$)	e^{-sT}	$s \in \mathbb{C}$

Fourier transform:

The Fourier transform and inverse Fourier transforms are given by:

$$\mathcal{F}\{f(t)\}(\omega) = F(\omega) := \int_{-\infty}^{\infty} f(t)e^{-j\omega t} dt, \qquad f(t) = \mathcal{F}^{-1}\{F(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t} d\omega.$$

Convolution:

$$f * g(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)\, d\tau.$$

Properties of the Fourier transform: $\mathcal{F}\{f(t)\} = F(\omega)$ in the following table:

$\mathcal{F}\{e^{j\theta t}f(t)\} = F(\omega - \theta)$	frequency shift
$\mathcal{F}\{f(t-T)\} = e^{-j\omega T}F(\omega)$	time shift
$\mathcal{F}\{f^{(n)}(t)\} = (j\omega)^n F(\omega)$	differentiation
$\mathcal{F}\{F(t)\} = 2\pi f(-\omega)$	symmetry
$\mathcal{F}\{f(at)\} = \frac{1}{\lvert a\rvert}F(\frac{\omega}{a})$	scaling
$\mathcal{F}\{f * g(t)\} = \mathcal{F}\{f(t)\}\mathcal{F}\{g(t)\}$	convolution

Table of standard Fourier transforms:

$f(t)$	$\mathcal{F}\{f(t)\}(\omega)$
$e^{-a\lvert t\rvert}$ (for $a > 0$)	$\frac{2a}{a^2+\omega^2}$
$\text{rect}_T(t)$	$\text{sinc}(\frac{T\omega}{2})$
1	$2\pi\delta(\omega)$
$\delta(t)$	1

Fourier series:

The Fourier series of a periodic function $f(t)$ with fundamental period T is given by

$$S[f] = \frac{a_0}{2} + \sum_{n=1}^{\infty}\Big(a_n\cos(\omega_n t) + b_n\sin(\omega_n t)\Big)$$

where

$$\omega_n = \frac{2\pi n}{T}, \qquad a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(\omega_n t)dt, \qquad b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(\omega_n t)dt.$$

Coordinate systems:

Cylindrical polar coordinates

$$(x, y, z) = (r\cos(\theta), r\sin(\theta), z)$$
$$(r, \theta, z) = \Big(\sqrt{x^2+y^2}, \arctan(\tfrac{y}{x}), z\Big)$$
$$dV = r dr d\theta dz.$$

Spherical polar coordinates

$$(x, y, z) = (\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi))$$
$$(\rho, \theta, \phi) = \Big(\sqrt{x^2+y^2+z^2}, \arctan(\tfrac{y}{x}), \arccos(\tfrac{z}{\rho})\Big)$$
$$dV = \rho^2\sin(\phi) d\rho d\phi d\theta.$$