Data provided: Formula sheet

The University Of Sheffield.

MAS248

SCHOOL OF MATHEMATICS AND STATISTICS — **Autumn Semester 2013–14**

MATHEMATICS III (CHEMICAL) — **2 hours**

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) Find $\dfrac{dz}{dx}$, where $z = x^3 y + \sin 2x$ and $y = \ln x$. ***(7 marks)***

(ii) Find $\dfrac{dy}{dx}$, given that

$$\cos^2 x + \cos^2 y = \cos(2x + 2y).$$

(9 marks)

(iii) Write down the iteration formula for the Newton-Raphson method. Starting from $x_0 = 1.0$ use the Newton-Raphson method to find an approximation to a root of the equation

$$x^3 - x - 1 = 0,$$

correct to four decimal places. ***(9 marks)***

2 (i) Find and classify the stationary points of the function

$$xy^2 - x^2 - 2y^2 = 0.$$

(10 marks)

(ii) A periodic function, $f(x)$, of period 2π is defined by

$$f(x) = \pi^2 - x^2 \quad \text{for} \quad -\pi \le x < \pi.$$

Supposing that $f(x)$ has a convergent trigonometric Fourier series, show that

$$\pi^2 - x^2 = \frac{2\pi^2}{3} + \sum_{n=1}^{\infty} \frac{-4}{n^2}(-1)^n \cos nx.$$

(15 marks)

3 (i) The vector field, $\mathbf{F}$, is given by

$$\mathbf{F} = (2x + yz, xz, yx).$$

(a) Verify that $\nabla \times \mathbf{F} = \mathbf{0}$. ***(3 marks)***

(b) Find a scalar potential, V, such that $\mathbf{F} = \nabla V$. ***(10 marks)***

(ii) A vector field, $\mathbf{A}$, is given by

$$\mathbf{A} = (xy, 2xz^2, 3)$$

and a scalar field, u, is given by

$$u(x, y, z) = xyz.$$

(a) Evaluate $\nabla \cdot \mathbf{A}$ at the point with co-ordinates $(1, -3, 2)$. ***(3 marks)***

(b) Verify that

$$\nabla \times (u\mathbf{A}) = u(\nabla \times \mathbf{A}) - \mathbf{A} \times \nabla u.$$

(9 marks)

4 Show that the partial differential equation

$$\frac{\partial^2 y}{\partial t^2} - 9\frac{\partial^2 y}{\partial x^2} = 0,$$

has solutions of the form $y = f(x + \lambda t)$ for arbitrary functions f provided that $\lambda = -3$ or $\lambda = 3$. ***(6 marks)***

Give an interpretation, including a clear diagram, of the form of the solution in each case. ***(6 marks)***

Derive the solution that satisfies the conditions

$$y(x, 0) = x^2,$$

$$\frac{\partial y}{\partial t}(x, 0) = \cos 3x.$$

(13 marks)

End of Question Paper

Formula Sheet

Fourier Series

Suppose that $f(x)$ is defined on the interval $-L \leq x \leq L$. The Fourier series for $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

where

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx, \qquad n = 0, 1, 2, ...,$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx, \qquad n = 1, 2, 3, ...$$

On the interval $0 \leq x \leq L$ the Fourier cosine series for $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}, \qquad a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx$$

and the Fourier sine series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \qquad b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

Gradient of a Scalar Field

The gradient of the scalar field $\phi(x, y, z)$ is given by

$$\nabla \phi = \text{grad } \phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right).$$

Chain Rule

1 If $z = f(x, y)$, where $x = x(t)$, $y = y(t)$, then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

2 If $z = f(x, y)$, where $x = x(u, v)$, $y = y(u, v)$, then

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}, \qquad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}.$$

3 If $z = f(u, v)$, where $u = u(x, y)$, $v = v(x, y)$, then

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}, \qquad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}.$$

Maxima and Minima

1 The function $f(x, y)$ has a stationary point at (x_0, y_0) if

$$f_x = f_y = 0 \quad \text{at} \quad (x_0, y_0).$$

2 At (x_0, y_0), the function $f(x, y)$ has:

(i) a minimum if

$$f_{xx}f_{yy} - f_{xy}^2 > 0 \quad \text{and} \quad f_{xx} > 0 \quad \text{at} \quad (x_0, y_0),$$

(ii) a maximum if

$$f_{xx}f_{yy} - f_{xy}^2 > 0 \quad \text{and} \quad f_{xx} < 0 \quad \text{at} \quad (x_0, y_0),$$

(iii) a saddle point if

$$f_{xx}f_{yy} - f_{xy}^2 < 0 \quad \text{at} \quad (x_0, y_0).$$