Data provided: Formula Sheet

MAS250

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2013–14**

Mathematics II (Materials) **2 hours**

*Marks will be awarded for answers to all questions in Section A, and for your best **THREE** answers to questions in Section B. Section A carries 40 marks, and the marks awarded to each question or section of question are shown in italics.*

Section A

A1 Find the solution of the equation

$$\frac{dy}{dx} + xy = \frac{e^{-\frac{1}{2}x^2}}{x^2}$$

for $x > 0$ which satisfies $y = 1$ when $x = 1$. ***(7 marks)***

A2 Find the general solution of the equation

$$9\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 4y = 0.$$ ***(5 marks)***

A3 If

$$f(x, y) = xy^2 - 2\sin(x + 3y)$$

and

$$x = \frac{s}{r}, \qquad y = rs,$$

use the chain rule to find $\frac{\partial f}{\partial r}$ and $\frac{\partial f}{\partial s}$, giving your answers in terms of r and s, and simplifying where possible. ***(10 marks)***

Find also $\frac{\partial^2 f}{\partial r^2}$ in terms of r and s. ***(3 marks)***

A4 The following table shows the wage bills (in millions of pounds) and the final points totals of 10 Premier League football clubs for 2007-08:

Club	Wages (x)	Points (y)
Arsenal	101.3	83
Aston Villa	50.4	60
Blackburn Rovers	39.7	58
Bolton Wanderers	39	37
Chelsea	149	85
Everton	44.5	65
Fulham	39.3	36
Liverpool	80	76
Manchester City	54.2	55
Manchester United	121.1	87

Calculate the means and variances of x and y, and also the covariance between x and y. ***(10 marks)***

It is assumed that x and y satisfy the linear relationship

$$y = a + b(x - \overline{x}), \qquad (*)$$

where $\overline{x}$ is the mean of x.

Calculate the least squares estimates of a and b, correct to 3 significant figures. State, giving reasons, whether you expect $(*)$ to give a good model. ***(5 marks)***

Section B

B1 (a) Find the general solution of the equation

$$y\frac{dy}{dx} + y^2 = 1 \qquad \text{for } |y| > 1.$$ ***(8 marks)***

(b) Find the solution of the equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 4y = e^{-4x},$$

given that $y = 0$ and $\frac{dy}{dx} = 1$ when $x = 0$. ***(12 marks)***

B2 (a) Find a vector normal to the surface $\phi = 2$ at the point A with coordinates $(2, 0, -1)$, where

$$\phi = x \sin y + y^2 + xz^2.$$

(5 marks)

Hence find the equation of the tangent plane to the surface at A. ***(2 marks)***

Find also the directional derivative of ϕ at A, in the direction $\mathbf{d} = (1, 2, 2)$. ***(2 marks)***

(b) A scalar field ψ and a vector field $\mathbf{u}$ are given by

$$\psi = x \sinh y + y \cos z + x \sin z, \qquad \mathbf{u} = (x^2y, y^2z, z^2x).$$

Verify that

$$\nabla \times \nabla\psi = \mathbf{0}$$

and

$$\nabla\cdot(\nabla\times\mathbf{u}) = 0.$$

(11 marks)

B3 A function $f(x) = x^2$ is defined on the interval $-1 \leq x \leq 1$.

(a) Show that $f(x)$ can be represented by the Fourier series

$$\frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(n\pi x).$$

[You may use the fact that $\cos(n\pi) = (-1)^n$.] ***(18 marks)***

(b) Sketch the function given by the above Fourier series on the interval $-3 \leq x \leq 3$. ***(2 marks)***

B4 A string of unstretched length $l > 0$ has its ends fixed at $x = 0$ and $x = l$. Its displacement $y(x, t)$ in the transverse direction satisfies the equation

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}$$

where c is a constant.

(a) Using separation of variables, show that the general solution can be written as

$$y(x,t) = \sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{l}\right)\left[D_n\cos\left(\frac{n\pi ct}{l}\right) + E_n\sin\left(\frac{n\pi ct}{l}\right)\right]$$

where D_n and E_n are constants. ***(14 marks)***

(b) If $y = 0$ and $\dfrac{\partial y}{\partial t} = 2c\sin\left(\dfrac{2\pi x}{l}\right)$ at $t = 0$ show that

$$y(x,t) = \frac{l}{\pi}\sin\left(\frac{2\pi x}{l}\right)\sin\left(\frac{2\pi ct}{l}\right).$$

(6 marks)

End of Question Paper

FORMULA SHEET

Trigonometry

$1+\tan^2\theta = \sec^2\theta$

$1+\cot^2\theta = \text{cosec}^2\theta$

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\cos(A-B) = \cos A\cos B + \sin A\sin B$

$\sin(A+B) = \sin A\cos B + \cos A\sin B$

$\sin(A-B) = \sin A\cos B - \cos A\sin B$

$\tan(A+B) = \dfrac{\tan A+\tan B}{1-\tan A\tan B}$

$\tan(A-B) = \dfrac{\tan A-\tan B}{1+\tan A\tan B}$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\cos 2\theta = 2\cos^2\theta - 1 = 1-2\sin^2\theta$

$a\cos\theta + b\sin\theta = R\cos(\theta-\alpha)$, where $R=\sqrt{(a^2+b^2)}$, $\cos\alpha = a/R$ and $\sin\alpha = b/R$

Hyperbolic Functions

$\sinh x = \dfrac{1}{2}(e^x - e^{-x})$

$\cosh x = \dfrac{1}{2}(e^x + e^{-x})$

$\cosh^2 x - \sinh^2 x = 1$

$\text{sech}^2 x + \tanh^2 x = 1$

$2\sinh x\cosh x = \sinh 2x$

$\cosh 2x = 2\cosh^2 x - 1 = 2\sinh^2 x + 1$

$\sinh^{-1} x = \ln\left[x+\sqrt{(1+x^2)}\right], \quad \text{all } x$

$\cosh^{-1} x = \ln\left[x+\sqrt{(x^2-1)}\right], \quad x \geqslant 1$

$\tanh^{-1} x = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right), \quad |\, x \,| < 1$

$\coth^{-1} x = \dfrac{1}{2}\ln\left(\dfrac{x+1}{x-1}\right), \quad |\, x \,| > 1$

Differentiation and Integration

Function	**Derivative**
x^n	nx^{n-1}
$\ln x$	$\frac{1}{x}$
e^x	e^x
$\tan x$	$\sec^2 x$
$\cot x$	$-\text{cosec}^2 x$
$\sec x$	$\sec x \tan x$
$\text{cosec} x$	$-\text{cosec} x \cot x$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\text{sech}^2 x$
$\coth x$	$-\text{cosech}^2 x$
$\text{sech} x$	$-\text{sech} x \tanh x$
$\text{cosech} x$	$-\text{cosech} x \coth x$
$\sin^{-1} x$	$\frac{1}{\sqrt{(1-x^2)}}$
$\cos^{-1} x$	$-\frac{1}{\sqrt{(1-x^2)}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\cot^{-1} x$	$-\frac{1}{1+x^2}$
$\sinh^{-1} x$	$\frac{1}{\sqrt{(x^2+1)}}$
$\cosh^{-1} x$	$\frac{1}{\sqrt{(x^2-1)}}$
$\tanh^{-1} x$	$\frac{1}{1-x^2}, \quad \mid x \mid < 1$
$\coth^{-1} x$	$-\frac{1}{x^2-1}, \quad \mid x \mid > 1$

Function	**Integral**
$\dfrac{1}{a^2+x^2}$	$\dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)$
$\dfrac{1}{a^2-x^2}$	$\dfrac{1}{a}\tanh^{-1}\left(\dfrac{x}{a}\right)$
$\dfrac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\dfrac{x}{a}\right)$
$\dfrac{1}{\sqrt{a^2+x^2}}$	$\sinh^{-1}\left(\dfrac{x}{a}\right)$
$\dfrac{1}{\sqrt{x^2-a^2}}$	$\cosh^{-1}\left(\dfrac{x}{a}\right)$

Differentiation and Integration Formulae

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\int_a^b uv dx = [u \times (\text{integral of } v)]_a^b - \int_a^b \frac{du}{dx} \times (\text{integral of } v) dx$$

Partial Differentiation

Chain Rule

1. Suppose that $z = f(x, y)$ and that x and y are functions of t, i.e., $x = x(t)$, $y = y(t)$. Then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

2. Suppose that $z = f(x, y)$ and that x and y are functions of the variables r and s, i.e., $x = x(r, s)$, $y = y(r, s)$. Then

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}, \qquad \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

First-Order Differential Equations

1. **Direct Integration**

$$\frac{dy}{dx} = f(x)$$

$$y = \int f(x)dx + C$$

2. **Separation of Variables**

$$\frac{dy}{dx} = f(x)g(y)$$

$$\int \frac{dy}{g(y)} = \int f(x)\, dx$$

3. **Homogeneous Equations**

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

make the substitution $y = zx$ to give

$$z + x\frac{dz}{dx} = f(z)$$

4. **Linear Equations**

$$\frac{dy}{dx} + P(x)y = Q(x)$$

multiply both sides by the integrating factor $e^{\int P(x)dx}$ to give

$$\frac{d}{dx}\left(ye^{\int P(x)dx}\right) = Q(x)e^{\int P(x)dx}$$

The Second-Order Differential Equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

where a, b, and c are constants.

General solution is

$$y = \text{Complementary Function} + \text{Particular Integral}$$

The solution, y_c, is given by

(i) $y_c = Ae^{m_1 x} + Be^{m_2 x}$, if m_1 and m_2 real and different,

(ii) $y_c = e^{mx}(A + Bx)$, if m_1 and m_2 real and equal $(m_1 = m_2 = m)$,

(iii) $y_c = e^{px}(A\cos qx + B\sin qx)$, if m_1 and m_2 are complex $(m_1 = p + iq, m_2 = p - iq)$, where m_1 and m_2 are the roots of the *auxiliary equation*

$$am^2 + bm + c = 0$$

Particular Integral, y_p

$f(x) = Ax^2 + Bx + C$ $\qquad y_p = ax^2 + bx + c$

$f(x) = Ae^{kx}$ $\qquad y_p = ae^{kx}$
when k is not one of the roots of the auxiliary equation

$f(x) = Ae^{kx}$ $\qquad y_p = axe^{kx}$
when k is one of the roots of the auxiliary equation

$f(x) = A\cos mx + B\sin mx$ $\qquad y_p = a\cos mx + b\sin mx$
when $\sin mx$ or $\cos mx$ is not part of the complementary function

$f(x) = A\cos mx + B\sin mx$ $\qquad y_p = x(a\cos mx + b\sin mx)$
when $\sin mx$ or $\cos mx$ is part of the complementary function

Fourier Series

Suppose that $f(x)$ is defined on the interval $-l \leqslant x \leqslant l$. The Fourier series for $f(x)$ is given by

$$f(x) \;=\; \frac{a_0}{2} \;+\; \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} \;+\; b_n \sin\frac{n\pi x}{l}\right) \;,$$

where

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,\mathrm{d}x \;,\quad n = 0,1,2,\dots,$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x \;,\quad n = 0,1,2,\dots .$$

On the interval $0 \leqslant x \leqslant l$ the Fourier cosine series for $f(x)$ is

$$f(x) \;=\; \frac{a_0}{2} \;+\; \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} \;,\quad a_n \;=\; \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,\mathrm{d}x$$

and the Fourier sine series is

$$f(x) \;=\; \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \;,\quad b_n \;=\; \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x \;.$$

Vector Calculus

The gradient of the scalar field $\phi(x, y, z)$ is given by

$$\nabla\phi \;=\; \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) \;.$$

The divergence of a vector field $\mathbf{u}(x, y, z) = (u, v, w)$ is given by

$$\nabla\cdot\mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

The curl of a vector field $\mathbf{u}(x, y, z) = (u, v, w)$ is given by

$$\nabla\times\mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

The Laplacian ∇^2 is given by

$$\nabla^2 \;=\; \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Statistics

For data values $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$

Means $\quad \bar{x} = \dfrac{1}{n}\sum_{i=1}^{n} x_i \quad$ etc.

Variances $\quad s_x^2 = \dfrac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 = \dfrac{1}{n}\sum_{i=1}^{n}(x_i^2) - \bar{x}^2 \quad$ etc.

s_x is standard deviation

Covariance $\quad \mathrm{cov}(x, y) = \dfrac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) = \dfrac{1}{n}\sum_{i=1}^{n}(x_i y_i) - \bar{x}\bar{y}$

Correlation coefficient $\quad r = \dfrac{\mathrm{cov}(x, y)}{s_x s_y}$

Linear regression by least squares

The least squares fit to the linear relationship

$$y = a + b(x - \bar{x})$$

is given by

$$a = \bar{y}, \qquad b = \frac{\mathrm{cov}(x, y)}{s_x^2}$$

The corresponding mean square residual is $s_y^2(1 - r^2)$.