Data provided: Formula sheet

The University Of Sheffield.

MAS252

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2014–15**

Further Civil Engineering Mathematics and Computing **2 hours**

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) If $f(x, y, z) = 2\dfrac{y}{x-z} - \sin\dfrac{xz}{y}$ and

$$x = \frac{3t}{t^2+1}, y = e^{2t-2}, z = \ln t, \quad (t > 0),$$

use the chain rule to evaluate df/dt when $t = 1$. ***(14 marks)***

(ii) The power of an engine is given in terms of three parameters as

$$w(x, y, z) = \frac{x - 2z}{\sqrt{1+y^2}} - \ln\frac{x}{y}.$$

During the functioning of the engine the value of the parameter x increases from 1 to 1.2, the value of the parameter y decreases from 0.6 to 0.5 while the initial value of z is 0.3. Use the small error formula to find the variation of the third parameter, z, so that the power output of the engine does not change. Work with an accuracy of 2 decimal places. ***(11 marks)***

2 A function of x and y which satisfies Laplace's equation

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$$

is given in the form

$$(A \cos sx + B \sin sx)(C \cosh sy + D \sinh sy) \qquad (1)$$

where A, B, C and D are arbitrary constants and s is a parameter. If Laplace's equation is satisfied in the rectangular region $0 \leq x \leq a$, $0 \leq y \leq b$ and the boundary conditions on $x = 0$ and $x = a$ are given by $\Phi = 0$, use expression (1) to show that the the solution, $\Phi(x, y)$, of Laplace's equation which satisfies these conditions is

$$\Phi(x, y) = \sum_{n=1}^{\infty} \left(c_n \cosh \frac{n\pi y}{a} + d_n \sinh \frac{n\pi y}{a} \right) \sin \frac{n\pi x}{a}$$

where c_n, d_n are constants.
Given that on $y = 0$ the boundary condition is $\Phi = 1$, determine the constants c_n. ***(12 marks)***

With the value of c_n determined before, find the constants d_n provided $\Phi = x$ on $y = b$. ***(13 marks)***

3 (i) Find the Fourier series decomposition of the function $f(x) = x^2 - x$ in the interval $-\pi \leq x \leq \pi$. ***(15 marks)***

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(ii) With the series decomposition obtained at part (i) prove that in the case of $x = 0$ the equality

$$\pi^2 = 12 \left(1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \dots \right)$$

is valid. ***(4 marks)***

(iii) Values of y at $x = 2$ determined using the fourth-order Runge-Kutta method with two different step-lengths are given in the following table

h	y(2)
0.2	3.40978
0.4	3.39278

Use this data to estimate a value for h which will ensure that the error in the calculated value of $y(2)$ using a fourth-order Runge-Kutta method does not exceed 10^{-4}. Give your answer correct to 4 decimal places.

(6 marks)

4 (i) The function $y(x)$ satisfies the differential equation

$$\frac{dy}{dx} = x + \sin y + 0.6,$$

and the condition $y = 1.2$ when $x = 0$. Obtain the first **four** terms of the Taylor series solution of this equation, working correct to four decimal places. **NOTE**: Remember to put your calculator in *radian* mode when calculating $\sin y$. ***(10 marks)***

(ii) An explicit approximation to the heat conduction equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad u = u(x,t), \quad 0 < x < 1$$

is given (in the usual notations) by

$$u_{i,j+1} = ru_{i-1,j} + (1-2r)u_{i,j} + ru_{i+1,j}, \quad u_{i,j} = u(x_i, t_j)$$

where $r = k/h^2$. If the initial and boundary conditions associated with the heat conduction equation are given by

$$u(x,0) = 2x^2, \quad 0 < x < 1$$

$$u(0,t) = 0, \quad \text{and} \quad u(1,t) = 2, \quad t > 0$$

use the above explicit scheme, with $h = 0.2$ and $k = 0.008$ to calculate grid-points values of u at the first time step. Work correct to 3 decimal places. ***(7 marks)***

(iii) Show that the equation

$$x^2 - e^x - 6 = 0$$

has a root in the interval $(-2.6, -2.2)$ and perform *five* iterations of the bisection method to obtain a refined estimate of the interval which contains the root. Work correct to three decimal places. ***(8 marks)***

End of Question Paper

Formula sheet

- The local truncation error in the case of the 4th order Runge-Kutta method is given by
$$Y(x) - y(x) = Ch^4$$
where $Y(x)$ is the exact value, $y(x)$ is the estimated numerical value, C is a constant and h is the step size used in the numerical scheme.

- **Chain rule**

If $z = f(x, y)$, where x and y are both functions of t, so that $x = x(t)$ and $y = y(t)$ we have
$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$
If $z = f(x, y)$ and both x and y are functions of u and v, so that $x = x(u, v)$ and $y = y(u, v)$ then we have
$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$
$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

- **Fourier series**

If the function $f(x)$ is defined over the interval $-l \le x \le l$, then the Fourier series of $f(x)$ is
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$
where
$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\ dx, \quad (n = 0, 1, 2, \ldots)$$
$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\ dx \quad (n = 1, 2, 3, \ldots)$$
If the function $f(x)$ is defined over the interval $0 \le x \le l$, then the Fourier cosine series of $f(x)$ is
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}, \qquad a_n = \frac{2}{l}\int_{0}^{l} f(x)\cos\frac{n\pi x}{l}\ dx, \quad (n = 0, 1, 2, \ldots)$$
while the sine series of $f(x)$ is
$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}, \qquad b_n = \frac{2}{l}\int_{0}^{l} f(x)\sin\frac{n\pi x}{l}\ dx \quad (n = 1, 2, 3, \ldots)$$