Data provided: Formula sheet

MAS253

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2013-2014**

Mathematics for Engineering Modelling **2 hours**

Answer ***four*** *questions. If you answer more than four questions, only your best four will be counted.*

1 (i) Find an expression for the sum of the infinite series

$$4 + 12x + 36x^2 + 108x^3 + 324x^4 + ...$$

and determine its radius of convergence, R. Hence find the sum of the infinite series

$$4x + 6x^2 + 12x^3 + 27x^4 + ...$$

(8 marks)

(ii) Find the Maclaurin series for $\sin x$ and $\sinh x$ up to terms involving x^5. Use your expressions to find a relationship between $\sinh(\mathrm{i}\theta)$ and $\sin\theta$.
Find the Taylor series for the function $\sqrt{x}$ about the point $x = 4$, up to the term involving x^2. Use your result to find an approximate value for $\sqrt{4.5}$. Give your answer to 5 significant figures.

(10 marks)

(iii) Use l'Hôpital's rule to evaluate

(a) $$\lim_{x\to 0} \frac{\mathrm{e}^x + \mathrm{e}^{-x} - 2}{2\cos 2x - 2},$$

(b) $$\lim_{x\to 0} \left(\frac{1}{\sin x} - \frac{1+x}{x}\right).$$

(7 marks)

2 Consider the function

$$f(x) = \begin{cases} 0 & -1 < x < 0 \\ 1 - x & 0 \le x < 1 \end{cases}$$

(i) Sketch the Fourier series $F(x)$ of the function $f(x)$ for the range $-3 < x < 3$. ***(4 marks)***

(ii) Show that

$$F(x) = \frac{1}{4} + \frac{2}{\pi^2} \sum_{m=0}^{\infty} \frac{\cos(2m+1)\pi x}{(2m+1)^2} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin n\pi x}{n} .$$

(16 marks)

(iii) State Fourier's Theorem and deduce that

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + ...$$

(5 marks)

3 (i) Let $F(s)$ be the Laplace transform of $f(t)$. Calculate *by integration* the Laplace transform of each the following functions:

(a) $f(t) = \mathrm{e}^{-4t}(1 + 3t)\,,$

(b) $f(t) = u(t-2)\,t\,,$

where $u(t)$ is the unit Heaviside function. ***(7 marks)***

(ii) With the aid of the Table of Laplace transforms, find the inverse Laplace transforms of the following functions of s:

(a) $$\frac{2s-1}{(s-1)(s+1)(s+3)}\,,$$

(b) $$\frac{\mathrm{e}^{-3s} s}{s^2 - 4s + 20}\,.$$

(10 marks)

(iii) Use the method of Laplace transforms to solve the second-order ordinary differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}\,t^2} - 4\,y = \sin 2t,$$

subject to the initial conditions $y(0) = 1$ and $y'(0) = -2$.

(8 marks)

4 The temperature along a slim metal bar of length l diffuses according to the differential equation

$$\frac{\partial \phi}{\partial t} = \kappa \frac{\partial^2 \phi}{\partial x^2} .$$

where κ is a real positive constant.

(i) Using the *method of separation of variables*, show that if $\phi \to 0$ as $t \to \infty$, then solutions are of the form

$$\phi(x,t) = \mathrm{e}^{-\beta t}(A\cos\lambda x + B\sin\lambda x) ,$$

where β is a real positive constant. ***(13 marks)***

(ii) Find the general solution that satisfies the boundary conditions $\phi = 0$ at $x = 0$ (fixed zero temperature) and $\partial\phi/\partial x = 0$ at $x = l$ (insulated). ***(6 marks)***

(iii) Initially, the temperature distribution along the bar is $\phi = 5\sin(3\pi x/(2l))$. Find $\phi(x,t)$ for this particular case. ***(6 marks)***

5 (i) Evaluate the integral

$$\int_1^4 \int_1^{\sqrt{x}} (2y - 5x)\, \mathrm{d}y\, \mathrm{d}x .$$

(7 marks)

(ii) Sketch the region of integration for the double integral

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \frac{x+y}{\sqrt{x^2+y^2}}\, \mathrm{d}y\, \mathrm{d}x .$$

Using a change of coordinates, evaluate the integral. ***(9 marks)***

(iii) Sketch the region R, trapped by the conditions $y \le x$, $y \le 2 - x$ and $y \ge 0$. Using only one double-integral over the region R, evaluate

$$\int\int_R \frac{1}{1 + 2y - y^2}\, \mathrm{d}x\, \mathrm{d}y .$$

(9 marks)

End of Question Paper

For use with MAS253 first semester examination

Formulae for use in L2 Mechanical Engineering Mathematics Examination

These results may be quoted without proof unless proofs are asked for in the question.

Trigonometry

$$\sin 2P = 2\sin P \cos P,$$

$$\cos 2P = \cos^2 P - \sin^2 P = 2\cos^2 P - 1 = 1 - 2\sin^2 P,$$

$$\cos P \cos Q = \frac{1}{2}\{\cos(P+Q) + \cos(P-Q)\},$$

$$\sin P \sin Q = -\frac{1}{2}\{\cos(P+Q) - \cos(P-Q)\},$$

$$\sin P \cos Q = \frac{1}{2}\{\sin(P+Q) + \sin(P-Q)\}.$$

Geometric progression

The partial sum to n terms of

$$a + ar + ar^2 + \ldots\ldots + ar^{n-1} + \ldots\ldots$$

is

$$S_n = a(1-r^n)/(1-r), \quad r \neq 1 \quad .$$

Taylor Series for functions of one variable (x)

The Taylor series of $f(x)$ about $x = a$ is

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2!} f''(a)(x-a)^2 + \ldots.$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

The Maclaurin series of $f(x)$ is the special case of the Taylor series when $a = 0$:

$$f(x) = f(0) + f'(0)\, x + \frac{1}{2!} f''(0)\, x^2 + \ldots.$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

Important examples of Maclaurin series are:

$$e^x = 1 + x + \frac{1}{2!}x^2 + \ldots \qquad (R \text{ is infinite})$$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \ldots. \qquad (R \text{ is infinite})$$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \ldots. \qquad (R \text{ is infinite})$$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \ldots \qquad (R = 1)$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \ldots. \qquad (R = 1)$$

R is the radius of convergence.

Binomial Theorem

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 + \ldots..$$

If n is positive and integer, series terminates.

If n is negative or non-integer (or both), the series is an infinite series with the radius of convergence, $R = 1$.

Fourier Series

The Fourier series of $f(x)$ for $-l < x < l$ is

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right)$$

where $$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx \quad ,$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx\,, \quad n = 1, 2, \ldots$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx\,, \quad n = 1, 2, \ldots$$

Laplace Transform

The Laplace Transform of $f(t)$ is

$$F(s) = L(f(t)) = \int_0^\infty e^{-st} f(t)dt \quad .$$

For special cases, see later page.

Partial Differentiation

$$\delta F = F(x+\delta, y+\varepsilon) - F(x,y) \cong \delta \frac{\partial F}{\partial x} + \varepsilon \frac{\partial F}{\partial y}$$

Chain Rules:

1. Suppose that $F = F(x,y)$ and that x and y are functions of t, i.e. $x = x(t), y = y(t)$, then

$$\frac{dF}{dt} = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} \quad .$$

2. Suppose that $F = F(x,y)$ and that x and y are functions of the variables u and v, i.e. $x = x(u,v), y = y(u,v)$, then

$$\frac{\partial F}{\partial u} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial u}; \qquad \frac{\partial F}{\partial v} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial v} \quad .$$

Taylor Series for functions of two variables (x, y)

The Taylor series of $f(x,y)$ about $x = a, y = b$ is

$$f(x,y) = f(a,b) + \{(x-a)\, f_x(a,b) + (y-b)\, f_y(a,b)\} +$$

$$+ \frac{1}{2!}\left\{(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)\, f_{xy}(a,b) + \right.$$

$$\left. + (y-b)^2 f_{yy}(a,b)\right\} +$$

$$+ \ldots .$$

Here $f_x = \frac{\partial f}{\partial x}$ etc.

Partial Differential Equations (2 independent variables)

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$ Laplace's equation

$$\frac{\partial^2 V}{\partial x^2} = \frac{1}{K}\frac{\partial V}{\partial t}$$ Heat conduction (or diffusion) eqn.

equation

$$\frac{\partial^2 V}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2}$$ Wave equation

General Solution of ODEs

$$X'' = -\omega^2 X \Rightarrow X(x) = A\cos \omega x + B\sin \omega x$$

$$X'' = \omega^2 X \Rightarrow X(x) = C\cosh \omega x + D\sinh \omega x$$

or $Ee^{\omega x} + Fe^{-\omega x}$

$$T' = kT \Rightarrow T(t) = Ae^{kt}$$

Table of Laplace Transforms

$f(t)$	$F(s) = L(f(t))$
$f(t)$	$F(s)$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2 F(s) - sf(0) - f'(0)$
$f^{iv}(t)$	$s^4 F(s) - s^3 f(0) - s^2 f'(0) - sf''(0) - f'''(0)$
1	$1/s$
t	$1/s^2$
$t^{n-1}/(n-1)!\ (n \geq 1)$	$1/s^n$
e^{at}	$\frac{1}{s-a}$
$\frac{1}{a}\sin at$	$\frac{1}{s^2+a^2}$
$\cos at$	$\frac{s}{s^2+a^2}$
$\frac{1}{a}\sinh at$	$\frac{1}{s^2-a^2}$
$\cosh at$	$\frac{s}{s^2-a^2}$
$\frac{\sin at - at\cos at}{2a^3}$	$\frac{1}{(s^2+a^2)^2}$
$\frac{t\sin at}{2a}$	$\frac{s}{(s^2+a^2)^2}$
$e^{at} f(t)$	$F(s-a)$, where $F(s) = L(f(t))$
$\delta(t)$	1
$\delta(t-a)$	e^{-as}
$u(t-a)$	e^{-as}/s
$u(t-a)\ f(t-a)$	$e^{-as} F(s)$, where $F(s) = L(f(t))$