The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2013–14**

MAS271 Methods for Differential Equations **2 hours**

Attempt all the questions. The allocation of marks is shown in brackets.

1 Consider the system of equations

$$\dot{x} = -y - \frac{x^3}{2} + \frac{x^4}{2} + \frac{3}{2}y^4 - \frac{3}{4}y^5, \qquad \dot{y} = 4x - 6y^3 + x^4 + 3y^4 - x^5.$$

(i) Find the nature of the equilibrium point (0,0). ***(5 marks)***

(ii) If a suitable Liapunov function for the above system can be given by $V(x,y) = 4x^2 + y^2$, find $\dot{V}$ (or $\frac{dV}{dt}$). ***(5 marks)***

(iii) Verify that $\dot{V}$ can also be given by

$$-2(x-1)(y-2)(x^4+3y^4).$$

Is $V(x,y)$ a weak or a strong Liapunov function in a neighbourhood of the origin? ***(5 marks)***

(iv) Consider the following system

$$\dot{x} = -x$$
$$\dot{y} = 1 - x^2 - y^2.$$

Find the equilibrium points, classify them and sketch the phase portrait of the system. ***(10 marks)***

2 (i) Consider the equation

$$\dot{x} = 1 - 2(1+\mu)x + x^2,$$

where μ is a constant control parameter. Find the bifurcation points and plot the bifurcation diagram. ***(7 marks)***

(ii) Write the following ordinary differential equation

$$\frac{d^2x}{dt^2} + \alpha(1-x^2)\frac{dx}{dt} + x = 0,$$

where α is a positive constant, as a system of two first order equations in the form

$$\dot{x} = f(x,y), \quad \dot{y} = g(x,y),$$

i.e. find the forms of the functions f and g. ***(2 marks)***

A Liapunov function has the form $V(x,y) = ax^2 + by^2$ with positive a and b. What are the values of a and b such that the Liapunov function is a weak Liapunov function for $\alpha > 0$? ***(6 marks)***

(iii) Consider the system given by

$$\dot{x} = -2xe^{x^2+y^2}, \quad \dot{y} = -2ye^{x^2+y^2}.$$

Using the Liapunov function $V(x,y) = e^{x^2+y^2}$, state whether the equilibrium point (0,0) is stable, asymptotically stable or unstable. ***(3 marks)***

(iv) Determine the necessary and sufficient conditions for the function

$$V(x,y) = ax^2 + 2bxy + cy^2$$

to be positive definite. ***(4 marks)***

Is the following quadratic function positive definite, negative definite, or neither?

$$x^2 - xy - y^2.$$

(3 marks)

3 (i) Find the first three non-zero terms of two linearly independent power series solutions about $x = 0$ of

$$(2 + x^2)y'' - xy' + 4y = 0.$$

Deduce the radius of convergence. ***(11 marks)***

(ii) Consider the use of a Frobenius series solution about $x = 0$ for the equation

$$2xy'' - y' + x^2y = 0.$$

Show that the roots of indicial equation are 0 and 3/2. Do not follow through to solve the differential equation. ***(5 marks)***

(iii) Consider the equation

$$\dot{x} = (x^2 - \sigma).$$

Draw the bifurcation diagram for $\sigma > 0$ and discuss the equilibria. What happens at $\sigma = 0$? ***(5 marks)***

(iv) Solve the differential equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 0.$$

(4 marks)

4 (i) Solve

$$x\frac{d^2y}{dx^2} - \frac{dy}{dx} = 3x^2.$$

(5 marks)

(ii) For the following equation, find the equilibrium points, investigate the nature of stability at these points and sketch the phase portraits for $\alpha^2 > 1$ and $\alpha^2 = \frac{1}{2}$.

$$\ddot{\theta} = \sin\theta(\cos\theta - \alpha^2).$$

(20 marks)

End of Question Paper