The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Autumn Semester 2013–2014

Continuum Mechanics — 2 hours

*Answer **four** questions. You are advised **not** to answer more than four questions: if you do, only your best four will be counted.*

1 (i) Using the $\varepsilon - \delta$ relation

$$\varepsilon_{ijk}\varepsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

prove that

$$\boldsymbol{v} \times (\nabla \times \boldsymbol{v}) = \frac{1}{2}\nabla\big(\|\boldsymbol{v}\|^2\big) - (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}. \qquad (*)$$

(9 marks)

(ii) You are given that, in cylindrical coordinates r, ϕ, z, the gradient of a scalar field f, and the divergence and curl of a vector field $\boldsymbol{v} = (v_r, v_\phi, v_z)$ are given by

$$\nabla f = \frac{\partial f}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial f}{\partial \phi}\boldsymbol{e}_\phi + \frac{\partial f}{\partial z}\boldsymbol{e}_z,$$

$$\nabla\cdot\boldsymbol{v} = \frac{1}{r}\frac{\partial(rv_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

$$\nabla\times\boldsymbol{v} = \left(\frac{1}{r}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right)\boldsymbol{e}_r + \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right)\boldsymbol{e}_\phi + \frac{1}{r}\left(\frac{\partial(rv_\phi)}{\partial r} - \frac{\partial v_r}{\partial\phi}\right)\boldsymbol{e}_z$$

where $\boldsymbol{e}_r$, $\boldsymbol{e}_\phi$ and $\boldsymbol{e}_z$ are the unit vector of the cylindrical coordinate system.

(a) Show that vector field $\boldsymbol{v}$ with the components given by

$$v_r = \frac{1}{r}\frac{\partial\psi}{\partial z}, \quad v_\phi = 0, \quad v_z = -\frac{1}{r}\frac{\partial\psi}{\partial r}. \qquad (\dagger)$$

is divergence-free, i.e. $\nabla\cdot\boldsymbol{v} = 0$. ***(2 marks)***

(b) You are again given that vector field $\boldsymbol{v}$ is defined by ($\dagger$). In addition, you are given that function ψ is independent of ϕ. Using ($*$) express $(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}$ in terms of the function ψ. ***(14 marks)***

2 (i) Give the definition in words of the material derivative. Derive the expression for the material derivative of a function $f(\boldsymbol{x}, t)$ in the Eulerian description,

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \nabla f,$$

where $\boldsymbol{v}$ is the velocity. ***(7 marks)***

(ii) The motion of the continuum is described in Lagrangian variables by

$$x_1 = \xi_1 - \alpha\xi_2 \cosh \omega t, \quad x_2 = \xi_2 + \alpha\xi_1 \sinh \omega t, \quad x_3 = \xi_3,$$

where α and ω are positive constants.

(a) Calculate the Jacobian J of the transformation from $\boldsymbol{\xi}$ to $\boldsymbol{x}$. ***(6 marks)***

(b) Obtain the expression for the velocity field $\boldsymbol{v}$ in the Lagrangian and Eulerian variables. ***(12 marks)***

3 (i) Write down the expression for the surface traction, $\boldsymbol{t}$, in terms of the stress tensor, $\mathbf{T}$, and the unit normal to the surface, $\boldsymbol{n}$. Express it both in vector and coordinate form. ***(3 marks)***

(ii) Show that

$$\int_S \boldsymbol{t}\, dS = \int_V \nabla \cdot \mathbf{T}\, dV. \qquad (*)$$

where S is the surface of the volume V. ***(7 marks)***

(iii) Using the momentum equation written in the integral form,

$$\frac{d}{dt}\int_V \rho \boldsymbol{v}\, dV = \int_S \boldsymbol{t}\, dS + \int_V \rho \boldsymbol{b}\, dV,$$

and equation $(*)$, derive the momentum equation written in the differential form in Eulerian variables,

$$\rho \frac{D\boldsymbol{v}}{Dt} = \nabla \cdot \mathbf{T} + \rho \boldsymbol{b},$$

where ρ is the density, $\boldsymbol{v}$ the velocity, and $\boldsymbol{b}$ the body force. (You can use without derivation the formula for differentiation of an integral over a moving volume, $\frac{d}{dt}\int_V \rho f\, dV = \int_V \rho \frac{Df}{Dt}\, dV$, where f is an arbitrary function.) ***(5 marks)***

(iv) You are given that a continuum occupies the half-space $z > 0$ in Cartesian coordinates x, y, z. There is a constant body force in the negative z-direction, $\boldsymbol{b} = (0, 0, -g)$. You are also given that the stress tensor has the form $\mathbf{T} = -p\mathbf{I}$, where $\mathbf{I}$ is the unit tensor and p the pressure. The pressure p is proportional to the continuum density ρ, $p = \alpha\rho$, and $p = p_0 = \text{const}$ at $z = 0$. Determine the dependence of p on z. ***(10 marks)***

4 (i) Give the definition of ideal incompressible fluid. Use the mass conservation equation written in Lagrangian variables to show that the density of an incompressible fluid satisfies the equation

$$\frac{D\rho}{Dt} = 0.$$

Use the mass conservation equation written in Eulerian variables to show that the velocity of this fluid satisfies the equation

$$\nabla \cdot \boldsymbol{v} = 0.$$

When this fluid is called homogeneous? ***(10 marks)***

(ii) The fluid motion is called planar when, in Cartesian coordinates x, y, z, the z-component of the fluid velocity is zero while the x and y-component are independent of z. Show that the velocity of planar motion of an incompressible fluid can be expressed in terms of the flux function ψ as

$$v_x = \frac{\partial \psi}{\partial y}, \quad v_y = -\frac{\partial \psi}{\partial x}.$$

(7 marks)

(iii) Give the definition of a streamline. Show that streamlines of a planar motion of an incompressible fluid are defined by the equation

$$\psi(x, y) = \text{const}.$$

(You can use without derivation that the general equation of a streamline is $\dfrac{dx}{v_x} = \dfrac{dy}{v_y} = \dfrac{dz}{v_z}$.) ***(3 marks)***

(iv) The fluid motion is called potential when $\boldsymbol{v} = \nabla\varphi$, where φ is called the potential. Show that the flux function of a potential planar motion of an ideal incompressible fluid satisfies the Laplace equation

$$\nabla^2 \psi = 0.$$

(5 marks)

5 (i) In the linear elasticity the equilibrium of an isotropic material is described by the equation

$$(\lambda + \mu)\nabla(\nabla \cdot \boldsymbol{u}) + \mu\nabla^2\boldsymbol{u} + \rho\boldsymbol{b} = 0. \qquad (*)$$

You are given that, in cylindrical coordinates r, ϕ, z, the displacement vector $\boldsymbol{u}$ has only the ϕ-component and it is independent of ϕ, $\boldsymbol{u} = (0, u(r,z), 0)$. In addition, there is no body force, $\boldsymbol{b} = 0$. Show that, in this case, equation $(*)$ reduces to

$$\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial(ru)}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0. \qquad (\dagger)$$

You can use without proof the formulae

$$\nabla \cdot \boldsymbol{u} = \frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\phi}{\partial \phi} + \frac{\partial u_z}{\partial z},$$

$$\nabla^2\boldsymbol{u} = \nabla(\nabla \cdot \boldsymbol{u}) - \nabla \times \nabla \times \boldsymbol{u},$$

$$\nabla \times \boldsymbol{u} = \left(\frac{1}{r}\frac{\partial u_z}{\partial \phi} - \frac{\partial u_\phi}{\partial z}\right)\boldsymbol{e}_r + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\boldsymbol{e}_\phi + \frac{1}{r}\left(\frac{\partial(ru_\phi)}{\partial r} - \frac{\partial u_r}{\partial \phi}\right)\boldsymbol{e}_z.$$

(6 marks)

(ii) There is an elastic rod of radius R and length L. The rod axis coincides with the z-axis of cylindrical coordinates. Its ends are fixed at the planes $z = 0$ and $z = L$. Initially there are no stresses in the rod. Then the upper plane is rotated by a small angle α, so the displacement at $z = L$ is $\boldsymbol{u} = (0, \alpha r, 0)$. The lower plane does not move, so the displacement is zero at $z = 0$. You can assume that the displacement in the rod has the form $\boldsymbol{u} = (0, u(r,z), 0)$, so it is described by equation $(\dagger)$. Looking for the solution to this equation in the form $u = rf(z)$, determine the function $f(z)$. ***(5 marks)***

(iii) Use the expression of the stress tensor in terms of the displacement,

$$\mathbf{T} = \lambda\mathbf{I}\nabla \cdot \boldsymbol{u} + \mu\left[\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T\right],$$

where $\mathbf{I}$ is the unit tensor to calculate the stress tensor inside the rod. You can use without proof that, for $\boldsymbol{u} = (0, u(r,z), 0)$,

$$\nabla\boldsymbol{u} = \frac{\partial u}{\partial r}\boldsymbol{e}_r\boldsymbol{e}_\phi - \frac{u}{r}\boldsymbol{e}_\phi\boldsymbol{e}_r + \frac{\partial u}{\partial z}\boldsymbol{e}_z\boldsymbol{e}_\phi.$$

(5 marks)

(iv) Use the obtained expression for $\mathbf{T}$ to calculate the surface traction at the upper edge of the rod, $z = L$. Then determine the moment of the force applied to the upper plane that is needed to rotate it by the angle α. ***(9 marks)***

End of Question Paper