The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Autumn Semester 2013–2014

**Introduction to Relativity** — **2 hours**

*Answer **four** questions. You are advised **not** to answer more than four questions: if you do, only your best four will be counted.*

**1** (i) Two inertial frames, $R:(ct,x)$ and $\tilde{R}:(c\tilde{t},\tilde{x})$ are related by the Lorentz transformation

$$\begin{pmatrix} c\tilde{t} \\ \tilde{x} \end{pmatrix} = \gamma(u)\begin{pmatrix} 1 & -u/c \\ -u/c & 1 \end{pmatrix}\begin{pmatrix} ct \\ x \end{pmatrix},$$

where

$$\gamma(u) = \left(1 - \frac{u^2}{c^2}\right)^{-1/2}.$$

Two explosions occur on the $x$-axis of the reference frame $R$, one at $x=0$ and $t=0$, and the other at $x=L>0$ and $t=T>0$. An observer is travelling in the reference frame $\tilde{R}$. He sees that the two explosions occur at the same time.

(a) Calculate $u$. ***(4 marks)***

(b) What condition must be imposed on $L$ and $T$ in order that the observer can see the two explosions simultaneously? ***(2 marks)***

(ii) Now you are given that the observer travelling in the reference frame $\tilde{R}$ sees that the two explosions occur at the same position.

(a) Calculate $u$ in this case. ***(4 marks)***

(b) What condition must be imposed on $L$ and $T$ in order that the observer can see that the two explosions occur at the same position? ***(2 marks)***

(iii) There are three inertial frames, $R:(ct,x)$, $R_1:(ct_1,x_2)$, and $R_2:(ct_2,x_2)$. The frame $R_1$ travels with the velocity $u$ in the $x$-direction relative to the frame $R$, and the frame $R_2$ travels with the velocity $v$ in the $x$-direction relative to the frame $R_1$.

 Question 1 continued on next page

**1** (continued)

(a) Show that the frame $R_2$ travels with the velocity $w$ in the $x$-direction relative to the frame $R$, where

$$w = \frac{u + v}{1 + uv/c^2}.$$

***(10 marks)***

(b) A train is moving with the speed 100 m/s. A passenger is running along the train in the direction of the train motion with the speed 5 m/s. In accordance with the Newtonian mechanics the passenger speed with respect to the rest frame $R$ is 105 m/s. What is the relativistic correction to the passenger speed? ***(3 marks)***

**2** (i) Consider the Lorentz transformation with the matrix $L$ given by

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & A_{11} & A_{12} & A_{13} \\ 0 & A_{21} & A_{22} & A_{23} \\ 0 & A_{31} & A_{32} & A_{33} \end{pmatrix}.$$

Show that the matrix $A$ given by

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

is orthogonal, i.e. $A^{-1} = A^T$, where the superscript '$T$' indicates the transposition. Show that $\det(A) = \pm 1$. Also show that the Lorentz transformation is proper if the matrix $A$ is proper, i.e. if $\det(A) = 1$.

Note: You can use without proof the rule for multiplication of block matrices. If

$$M = \begin{pmatrix} N & 0 \\ 0 & S \end{pmatrix}, \quad M_1 = \begin{pmatrix} N_1 & 0 \\ 0 & S_1 \end{pmatrix},$$

where $N$ and $N_1$ are $m \times m$ matrices, and $S$ and $S_1$ are $n \times n$ matrices, then

$$MM_1 = \begin{pmatrix} NN_1 & 0 \\ 0 & SS_1 \end{pmatrix}$$

***(10 marks)***

(ii) (a) In a reference frame $R$ a particle is moving in the $xy$-plane with the speed $v$ at the angle $30^\circ$ with respect to the positive $x$-axis. Calculate the four-velocity of this particle. ***(3 marks)***

(b) Now you are given that an observer is in the reference frame $\tilde{R}$ that is moving with the velocity $u$ in the $x$-direction with respect to the reference frame $R$. The $\tilde{x}$, $\tilde{y}$ and $\tilde{z}$-axis of the reference frame $\tilde{R}$ are parallel to the $x$, $y$ and $z$-axis respectively of the reference frame $R$, so the coordinates in the two reference frames are related by the Lorentz transformation

$$\begin{pmatrix} c\tilde{t} \\ \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} \gamma(u) & -u\gamma(u)/c & 0 & 0 \\ -u\gamma(u)/c & \gamma(u) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}.$$

You are also given that the observer sees the particle moving at the angle $60^\circ$ with respect to the positive $\tilde{x}$-axis in his reference frame. Determine $v$ if $u = c/2$. ***(12 marks)***

**3** (i) Give the definition of timelike, spacelike and null four-vectors. ***(3 marks)***

(ii) (a) The four-vector $X$ is timelike. In the reference frame $R$ it is given by $X = (x^0, x^1, 0, 0)$. Show that there is a Lorentz transformation

$$\begin{pmatrix} \tilde{x}^0 \\ \tilde{x}^1 \end{pmatrix} = \gamma(u) \begin{pmatrix} 1 & -u/c \\ -u/c & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \end{pmatrix} \qquad (*)$$

such that, in the reference frame $\tilde{R}$, $\tilde{X} = (\alpha, 0, 0, 0)$, where $\alpha^2 = g(X, X)$. ***(6 marks)***

(b) The four-vector $X$ is spacelike. In the reference frame $R$ it is given by $X = (x^0, x^1, 0, 0)$. Show that there is a Lorentz transformation $(*)$ such that, in the reference frame $\tilde{R}$, $\tilde{X} = (0, \beta, 0, 0)$, where $\beta^2 = -g(X, X)$. ***(4 marks)***

(c) The four-vector $X$ is null. In the reference frame $R$ it is given by $X = (x^0, x^1, 0, 0)$. Show that there is such a Lorentz transformation $(*)$ such that, in the reference frame $\tilde{R}$, either $\tilde{X} = (1, 1, 0, 0)$, or $\tilde{X} = (-1, 1, 0, 0)$. ***(8 marks)***

(d) A timelike four-vector $X = (x^0, x^1, 0, 0)$ is future-pointing in the reference frame $R$. Show that it is future-pointing in any other reference frame $\tilde{R}$ related to $R$ by the Lorentz transformation $(*)$. ***(4 marks)***

**4** (i) In an inertial frame $R$ an observer has wordline $X(t) = (ct, x(t), 0, 0)$. Define the proper time, four-velocity $V$ and four-acceleration $A$ for this observed. ***(4 marks)***

(ii) Show that the four-velocity and four-acceleration are given by

$$V = c(\cosh\rho, \sinh\rho, 0, 0), \quad A = c\frac{d\rho}{d\tau}(\sinh\rho, \cosh\rho, 0, 0),$$

and that

$$g(A, A) = -c^2\left(\frac{d\rho}{d\tau}\right)^2,$$

where the rapidity is defined by $\tanh\rho = v/c$. ***(5 marks)***

(iii) A particle is moving along the $x$-axis of an inertial frame $R$ in the positive $x$-direction. At the initial moment of time $t = 0$ its position is $x = 0$ and its velocity is $v = 0$. In the particle instantaneous rest frame, the acceleration felt by the particle is constant, in the $x$-direction, and equal to $a$. Show that the path of the particle in $R$ is given by the parametric equation

$$X(\tau) = \frac{c^2}{a}\left(\sinh\frac{a\tau}{c}, \cosh\frac{a\tau}{c} - 1\right).$$

(You can assume that the proper time of the particle $\tau = 0$ at $t = 0$).
***(7 marks)***

(iv) One of the two twin brothers stayed on the Earth. The second twin brother travelled in a spaceship that set out from the Earth to travel to a star. The ship accelerated with acceleration $a = g$ (the acceleration equal to the acceleration due to gravity on the Earth) for $T$ years. Then it decelerated for $T$ years with acceleration $a = -g$, at which point it arrived at the star.

The ship did not stop at the star and made a similar return journey. When the ship returned to the Earth, it turned out that the time passed on the Earth was 20 years. Calculate $T$. What was the age difference of the two brothers? (You can take that 1 years $\approx 3.15 \times 10^7$ s and $g \approx 10$ m s$^{-2}$).
***(9 marks)***

**5** (i) Define the rest mass and four-momentum of a particle. ***(3 marks)***

(ii) A proton of the rest mass $m_p$ is moving with speed $u = 0.99c$ in the positive direction of the $x$-axis in an inertial frame $R$. Another proton, also with the rest mass $m_p$, is moving with the same speed $u = 0.99c$ in the positive direction of the $y$-axis in the frame $R$. The two protons collide and two new particles are formed. One particle has the rest mass $2m_p$ and it is moving with speed $v = 0.9c$ in the positive $x$-direction of the frame $R$. The second particle has mass $M$ and it is moving with speed $w$ in the $xy$-plane at the angle $\alpha$ with respect to the $x$-axis. Using the law of the four-momentum conservation, show that $M$, $w$ and $\alpha$ are determined by the system of equations

$$2m_p\gamma(u) = 2m_p\gamma(v) + M\gamma(w),$$
$$m_p u\gamma(u) = 2m_p v\gamma(v) + Mw\gamma(w)\cos\alpha,$$
$$m_p u\gamma(u) = Mw\gamma(w)\sin\alpha.$$

Use this system of equations to calculate $M/m_p$, $w/c$ and $\alpha$. ***(22 marks)***

## End of Question Paper