The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2013–14**

Metric Spaces **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) Let $n \geqslant 1$. Define the taxicab metric d_1 and the Euclidean metric d_2 on $\mathbb{R}^n$. ***(2 marks)***

When $n = 4$, find the distance between $(2, 0, 1, 4)$ and $(1, 9, 8, 4)$ in each of these metrics. ***(2 marks)***

(a) Let $(a, b), (x, y) \in \mathbb{R}^2$ and let $r > 0$. Show that

$$d_1((x, y), (a, b)) \geqslant d_2((x, y), (a, b)).$$

Show also that

$$2d_2((x, y), (a, b))^2 - d_1((x, y), (a, b))^2 = (|x - a| - |y - b|)^2$$

and deduce that $\sqrt{2}d_2((x, y), (a, b)) \geqslant d_1((x, y), (a, b))$. ***(7 marks)***

Hence show that

$$B_1((x, y), r) \subseteq B_2((x, y), r) \subseteq B_1((x, y), \sqrt{2}r).$$

Sketch a diagram showing the boundaries of these three open balls when $(x, y) = 0$. ***(5 marks)***

(b) There are four points that are in the closed ball $B_2[(0, 0), r]$ but not in the open ball $B_1((0, 0), \sqrt{2}r)$. Identify the co-ordinates of these four points in terms of r. ***(2 marks)***

(ii) (a) Let $((a_n, b_n))$ be a sequence in $\mathbb{R}^2$ and let $(a, b) \in \mathbb{R}^2$. Show that $((a_n, b_n)) \to (a, b)$ under d_2 if and only if $((a_n, b_n)) \to (a, b)$ under d_1. ***(4 marks)***

(b) Let A be a subset of $\mathbb{R}^2$. Show that if A is compact under d_1 then A is compact under d_2. ***(3 marks)***

2 (i) Let A be a subset of a metric space (X, d).

(a) Explain what it means to say that A is *closed* and what it means to say that A is *open*. ***(4 marks)***

(b) Show that if the complement $X \backslash A$ is closed then A is open. ***(4 marks)***

(c) Let $A_1 = \{(x, y) \in \mathbb{R}^2 : 4 < x^2 + y^2 \leqslant 16\}$. Use the definitions of *closed* and *open* in (a) to show that A_1 is neither closed nor open. ***(4 marks)***

(ii) (a) Give two equivalent definitions of a *continuous* function $f : (X, d_X) \to (Y, d_Y)$ between metric spaces. One definition should be sequential and the other should involve open balls. ***(5 marks)***

(b) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function such that for all $(x, y) \in \mathbb{R}^2$, $f((x, y)) = x^2 + y^2$. Using the sequential definition of continuity, show that f is continuous when $\mathbb{R}^2$ has the metric d_2 and $\mathbb{R}$ has its usual metric. ***(3 marks)***

(c) Let

$$\begin{aligned} A_2 &= \{(x, y) \in \mathbb{R}^2 : 4 < x^2 + y^2 < 16\}, \\ A_3 &= \{(x, y) \in \mathbb{R}^2 : 4 \leqslant x^2 + y^2 \leqslant 16\}. \end{aligned}$$

Express A_2 and A_3 as inverse images of intervals in $\mathbb{R}$. Hence, or otherwise, show that A_2 is open and A_3 is closed. State clearly any result relating continuity to inverse images of open or closed sets that you use. ***(5 marks)***

3 The metrics d_1 and d_∞ on the space $C[0,1]$ of continuous functions from $[0,1]$ to $\mathbb{R}$ are given by the rules

$$d_1(f,g) = \int_0^1 |f(x)-g(x)|\,dx, \quad d_\infty(f,g) = \sup_{0\le x\le 1} |f(x)-g(x)| \text{ for all } f,g \in C[0,1].$$

(i) Let (x_n) be a sequence in a metric space (X,d). Explain what it means to say that (x_n) is a *Cauchy* sequence. Show that if (x_n) is convergent in (X,d) then it is Cauchy. ***(5 marks)***

(ii) For $n \ge 1$, let f_n be the function in $C[0,1]$ such that

$$f_n(x) = \begin{cases} 1 - 2^n x & \text{if } 0 \le x \le \dfrac{1}{2^n}; \\ 0 & \text{if } \dfrac{1}{2^n} \le x \le 1 \end{cases}$$

and let f be the zero function, $f(x) = 0$ for $0 \le x \le 1$.

(a) Sketch the graphs of f_2 and $f_1 - f_2$. ***(3 marks)***

(b) Find $d_1(f_n, f)$ and deduce that, under d_1, $f_n \to f$. Find also $d_\infty(f_n, f)$ and deduce that, under d_∞, $f_n \not\to f$. ***(5 marks)***

(c) Let $m > n$ and let $x = \dfrac{1}{2^{n+1}}$. Compute $f_n(x)$ and $f_m(x)$. Deduce that $d_\infty(f_m, f_n) \ge \dfrac{1}{2}$ and hence that no subsequence of (f_n) is Cauchy in $(C[0,1], d_\infty)$. ***(6 marks)***

Is (f_n) a Cauchy sequence in $(C[0,1], d_1)$? Justify your answer. ***(2 marks)***

(d) Use the sequence (f_n) to show that, in $(C[0,1], d_\infty)$, the subspace

$$A = \{g \in C[0,1] : 0 \le g(x) \le 1 \text{ for all } x \in [0,1]\}$$

is not compact. ***(4 marks)***

4 (i) Define a *contraction* on a metric space (X, d) and state, without proof, the Contraction Mapping Principle. ***(4 marks)***

Throughout the remainder of this question you may assume the following differential criterion: if $f\colon [a,b] \to [a,b]$ is differentiable on the closed bounded interval $[a,b]$ then f is a contraction on $[a,b]$ if and only if there exists $k < 1$ with $|f'(x)| \leqslant k$ for all $x \in (a,b)$.

(ii) Consider the following functions in $C[-\pi, \pi]$:

$$f_1(x) = \cos(x);\ f_2(x) = \cos(\cos(x));\ f_3(x) = \sin(\cos(x)),\ f_4(x) = \cos(\sin(x)).$$

You may assume that, for all $y \in [-1, 1]$, $|\sin(y)| < \sin(1) < 1$.

(a) Show that f_1 and f_3 are not contractions on $[-\pi, \pi]$. ***(3 marks)***

(b) Show that f_2 and f_4 satisfy the differential criterion with $k = \sin(1)$ and deduce that f_2 and f_4 each have a unique fixed point in $[-\pi, \pi]$. ***(6 marks)***

(iii) (a) Let (X, d) be a complete metric space, and let $f\colon X \to X$ have the property that, for some positive integer m, the iterate f^m is a contraction of X. Show that f has a unique fixed point in X. ***(4 marks)***

(b) Deduce from (a) and (ii)(b) that f_1 has a unique fixed point in $[-\pi, \pi]$. ***(1 mark)***

(c) Show that f_3^2 is a contraction and deduce that f_3 has a unique fixed point in $[-\pi, \pi]$. ***(4 marks)***

(d) Let x be the unique fixed point of f_3 in $[-\pi, \pi]$ and let y be the unique fixed point of f_4 in $[-\pi, \pi]$. Show that $y = \cos(x)$ and $x = \sin(y)$. ***(3 marks)***

End of Question Paper